THE REPAIRABLE QUEUING SYSTEM MIG (E_2 / H)|1 ON THE METHOD OF VECTOR MARKOV PROCESS

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ABSTRACT

By forming a vector Markov process, the repairable queuing system M|G (E_2/H)|1 is studied in this paper. We obtain some system characters, the reliability indices of the server, and the time distribution of a customer spent on the server.

INTRODUCTION

In classic queuing system, usually we suppose the service station would not lose efficiency or repair it immediately if it lose efficiency. But in the true queuing system, the service life act as some distribution and if it lose efficiency we can repair it, then it is necessary for us to study this kind of repairable queuing system Cao Jin Hua and Cheng Kan [1] have studied the repairable service station M|G|1 queuing system and in the research work of this paper we use vector Markove process method to study repairable queuing system M|G (E_2 / H)|1where E_2/H indicate life time distribution of the server is Erlangian is studied and the repairing time is a normal continuous distribution other makes are same as the common queuing system.

2. MODEL DEFINATION

2.1 Assumption

The interarrival time of successive customers are independently identically exponentially distributed with mean λ.

The service time of customer is common continuous distribution G(t) written as:

$$G(t) = \int_0^t g(x)\,dx = 1 - \exp\left\{-\int_0^t \mu(x)\,dx\right\},\ \int_0^t g(x)\,dx = \frac{1}{\mu} < \infty$$

(iii) Life time distribution of a server is Erlang distribution with parameter δ_1 δ_2, suppose the density of the distribution is,

$$f(t) = \frac{\delta_1 \delta_2}{\delta_1 - \delta_2} [\exp(-\delta_2 t) - \exp(-\delta_1 t)],\ t \geq 0,\ \delta_1 \neq \delta_2;$$

(iv) The repair time distribution of the server is a general continuous distribution.

$$H(t) = \int_0^t h(y)dy = 1\text{-}\exp\left\{-\int_0^t \beta(y)dy\right\},\ \int_0^\infty yh(y)\,dy = \frac{1}{\beta} < \infty$$

(v) Single service station every time service for one customer when the service station is be repairing the customer need waiting in the line.
(vi) The interarrival time of the customers, the time customer be serviced service time, repair time are all mutually independent.
(vii) Repair it as new when the service station get wrong.

2.2 Vector Markov Process

Suppose $S(t) = n$ indicate at time t queue length is n (including the customers who is being serviced), $n = 0, 1, 2, \ldots\ldots\ldots$; from suppose s(t) is not the continuous time Markov Chain. Lead in one replenish.
$I(t) = i$ indicate at time t service station is in i state,
$i = 1, 2$; use
$I(t) = 3$ indicate the service station lose its efficiency at the moment t.
$X(t) = x$ represent the elapsed service time of the customer currently being served and Y(t) represent the elapsed repair time of the failed server. Here $0 \leq x, y < \infty$
Thus the state space of the process $\{S(t), I(t), X(t), Y(t)\}$ is as

$$J = \{(0,1);, (n,i,x), (n,3,x,y)/i = 1,2;\ n = 1,2,\ldots;\ 0 \leq x, y < \infty\}$$

In which x and y - are values taken by x(t), y(t) respectively, and consequently we may define the following probabilities for any time t.

$$P_{0i}(t) = P\ s(t) = 0,\ I\left\{\ (t) = i,\ i = 1, 2;\ \right\}$$

$$P_{ni}(t,x)dx = P\ S(t)\left\{= n,\ I(t) = i,\ x \leq x(t) < x + dx\ ,\right\}$$
$$i = 1, 2; n = 1,2,\ldots\ldots;\quad 0 \leq x < \infty$$

$$P_{n3}(t,x,y)dy = P\left\{S(t) = n,\ I(t) = 3,\ x(t) = x,\ y \leq Y(t) < y+dy\ \right\}$$

$n = 1, 2, \ldots\ldots\ldots; 0 \le x < \infty$

The transition between the system states are shown in fig. 1

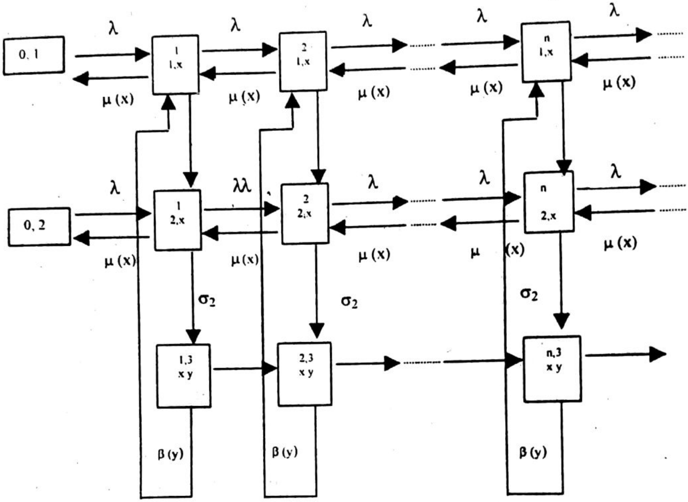


Fig. 1. State transition rate diagram for M|G (E_2/H)|1.

By the transition-rate diagram we can get the system of differential-integral equations.

$$\left(\frac{\delta}{\delta t} + \lambda\right) P_{0i}(t) = \int_0^\infty \mu(x)\, P_{1i}(t,x)dx, \quad i = 1,2; \qquad (1)$$

$$\left(\frac{\delta}{\delta t} + \frac{\delta}{\delta x} + \lambda + \sigma_1 + \mu(x)\right) P_{ni}(t,x) = \lambda P_{(n-1)i}(t,x) + \int_0^\infty \beta(y) P_{n3}(t,x,y)dy$$

$n = 1, 2, \ldots\ldots\ldots;$ (2)

$$\left(\frac{\delta}{\delta t} + \frac{\delta}{\delta x} + \lambda + \sigma_2 + \mu(x)\right) P_{n2}(t,x) = \sigma_1 P_{ni}(t,x) + \lambda P_{(n-1)2}(t,x)$$

$n = 1, 2, \ldots\ldots$ (3,

$$\left(\frac{\delta}{\delta t} + \frac{\delta}{\delta y} + \lambda + \beta(y)\right) P_{n3}(t,x,y) = \lambda P_{(n-1)3}(t,x,y).$$

$n = 1, 2, \ldots\ldots$ (4)

where $P_{01}(t, x) = P_{02}(t, x) = P_{03}(t, x) = 0$

The boundary conditions are:

$$P_{i1}(t, 0) = \lambda P_{01}(t)\, u(t) + \int_0^\infty \mu(x))P_{21}(t,x)dx, \quad i = 1, 2; \qquad (5)$$

$$P_{n1}(t, 0) \int_0^\infty \mu(x)\, P_{(n+1)1}(t,x)dx, \quad n = 2, 3, \ldots; \quad i = 1,2; \qquad (6)$$

$$P_{n3}(t,x,0) = \sigma_2 P_{n2}(t,x), \quad n = 1, 2\ldots.., \qquad (7)$$

Where $u(t) = \begin{cases} 0, & t = 0 \\ 1, & t > 0 \end{cases}$

The initial conditions are $P_{01}(0) = 1$, and others are zero.

3. The State Probabilities

3.1 Notation

For case, we take

$a = s + \lambda - \lambda z$, $b_1 = s + \lambda + \sigma_1 - \lambda z$, $i = 1,2$

$e_1 = [1,0]'$, $e_2 = [0,1]'$

$$\hat{f}(s) = \int_0^\infty \exp(-st)\, f(t)\, dt, \quad Re(s) \geq 0 \qquad \text{Laplace transform}$$

$$f(s) = \int_0^\infty \exp(-st)\, \underline{f}(t)\, dt, \quad Re(s) \geq 0 \qquad \text{Laplace-Stieltjes Transform}$$

$\bar{F}(t) = 1 - F(t)$

$P_{ni}(t, x) = \bar{G}(x)\, R_{n1}(t, x)$, $i = 1, 2$

$P_i(t, x, z) = \sum_{n=1}^{\infty} P_{n1}(t, x)\, z^n$, $i = 1, 2$ z-Transform

$$R_1(t, x, z) = \sum_{n=1}^{\infty} R_{n1}(t, x) z^n, \quad \text{---------- z-Transform}$$

$$P_3(t, x, z) = \sum_{n=1}^{\infty} P_{n3}(t, x, y) z^n, \quad \text{---------- z-Transform}$$

$$Q_{ni}(t, x) = \bar{G}(x) q_{ni}(t, x), \quad i = 1, 2;$$

$$Q_i(t, x, z) = \sum_{n=1}^{\infty} Q_{ni}(t, x) z^n, \quad i = 1, 2; \text{----- z-Transform}$$

$$q_i(t, x, z) = \sum_{n=1}^{\infty} q_{ni}(t, x) z^n, \quad i = 1, 2; \text{----- z-Transform}$$

$P(--) = [P_1(--), P_2(--)]'$, $Q(--) = [Q_1(--), Q_2(--)]'$, $R(--) = [R_1(--), R_2(--)]'$
$q(--) = [q_1(--), q_2(--)]'$, $r(--) = [r_1(--), r_2(--)]'$, $P_0(--) = [P_{01}(--), P_{02}(--)]'$
$q_0(--) = [q_{01}(--), q_{02}(--)]'$
$\varepsilon(z) = \sqrt{(\sigma_1 - \sigma_2)^2 + 4\sigma_1\sigma_2 h^*(s+\lambda-\lambda z)}$

$\tau_1 = \frac{1}{2}[\varepsilon(z) - 2a - (\sigma_1 - \sigma_2)]$, $\tau_1 = \lim_{z \to 1} \tau_1 = \frac{1}{2}\left[\sqrt{(\sigma_1 - \sigma_2)^2 + 4\sigma_1\sigma_2 h^3(s)} - 2s - \sigma_1 - \sigma_2\right.$

$\tau_2 = \frac{1}{2}\left[-\varepsilon(z) - 2a - (\sigma_1 - \sigma_2)\right]$, $\bar{\tau}_2 = \lim_{z \to 1} \tau_2 = \frac{1}{2}\left[-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\sigma_1\sigma_2 h^*(s)} - 2s - \sigma_1 - \sigma_2\right.$

$\sigma = (\sigma_1 - \sigma_2) / 2\sigma_1$, $\Delta\{x_1, x_2\} = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}$

$$S_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad s(x) \text{ ------ Dirac function, } \int_0^{\infty} s(x)dx = 1$$

3.2 Solution of the Set of Equations

Taking Laplace transform of eq. (1) to (7) we get

$$\left[s+\lambda\right]^{*}P01(s) = \delta_{11} + \int_0^{\infty} \mu(x)\, P^{*}_{1i}(s,x)\,dx, \qquad i = 1, 2 \quad (8)$$

$$\left[\frac{\partial}{\partial x} + s + \lambda + \sigma_1 + \mu(x)\right] P^{*}_{n1}(s,x) = \lambda P^{*}_{n-11}(s,x) + \int_0^{\infty} \beta(y) P^{*}_{n3}(s,x,y)\,dy, \quad (9)$$

$n = 1, 2, \ldots\ldots\ldots\ldots;$

$$\left[\frac{\partial}{\partial x} + s + \lambda + \sigma_2 + \mu(x)\right] P^{*}_{n2}(s,x) = \lambda P^{*}_{n-12}(s,x) + \sigma_1 P^{*}_{n1}(s,x), \quad (10)$$

$n = 1, 2, \ldots\ldots;$

$$\left[\frac{\partial}{\partial x} + s + \lambda + \beta(y)\right] P^{*}_{n3}(s,x,y) = \lambda P^{*}_{n-13}(s,x,y),\ n = 1,2,\ldots\ldots; \quad (11)$$

$$P^{*}_{1i}(s,0) = \lambda P^{*}_{01}(s) + \int_0^{\infty} \mu(x) P^{*}_{1i}(s,x)\,dx, \qquad n = 1,2,\ ,\ldots\ldots\ldots, \quad (12)$$

$$P^{*}_{n1}(s,0) = \int_0^{\infty} \mu(x) P^{*}_{n+1i}(s,x)\,dx, \qquad n = 2, 3, \ldots\ldots,\ i = 1,2\ldots (13)$$

$$P_{n3}(s,x,0) = \sigma_2 P^{*}_{n2}(s,x), \qquad n = 1, 2\ldots\ldots\ldots\ldots \quad (14)$$

Taking z-Transform $P^{*}_{ni}(s,x), P^{*}_{n3}(s,x,y)$, $(I = 1,2)$ from eq. (8) to(14)

We get

$$\left[\frac{\partial}{\partial x} + s + \lambda + \sigma_2 + \mu(x)\right] P^{*}_{1}(s,x,z) = \lambda z P^{*}_{1}(s,x,z) + \int_0^{\infty} \beta(y) P^{*}_{3}(s,x,z)\,dy \quad (15)$$

$$\left[\frac{\partial}{\partial x} + s + \lambda + \sigma_2 + \mu(x)\right] P^{*}_{2}(s,x,z) = \lambda z P^{*}_{2}(s,x,z) + \sigma_1 P^{*}_{1}(s,x,z) \quad (16)$$

$$\left[\frac{\partial}{\partial x} + s + \lambda + \sigma_2 + \beta(y)\right] P^{*}_{3}(s,x,y,z) = \lambda z P^{*}_{3}(s,x,y,z) \quad (17)$$

$$z P^{*}_{1}(s,0,z) = z\delta_{1i} - za P^{*}_{01}(s) + \int_0^{\infty} \mu(x)\, P^{*}_{i}(s,x,z)\,dx, \qquad i = 1, 2; \quad (18)$$

$$P^{*}_{3}(s,x,0,z) = \sigma_2 P^{*}_{2}(s,x,z) \quad (19)$$

From eq. (17). (19) we get

$$P^{*}_{3}(s,x,0,z) = \sigma_2\ \bar{H}(y)\, P^{*}_{2}(s,x,z)\, e^{-ay} \quad (20)$$

According to eq. (20) we obtained

$$\int_0^\infty \beta(y)\, P^*_3(s, x, y, z)\, dy = \sigma_2 P^*_2(s, x, z)\, h^3(a) \qquad (21)$$

From (15), (16), (18), (21) we get

$$\frac{\partial}{\partial x} R^*(s, x, z) = AR^*(s, x, z) \qquad (22)$$

$$zR^*(s, 0, z) = z\,[e_1 - a\, P^*_{0_1}(s)] + \int_0^\infty g(x)\, R^*(s, x, z)\, dx \qquad (23)$$

where $A = \begin{pmatrix} -a-\sigma_1 & \sigma_2 h^{*2}(a) \\ \sigma_1 & -a-\sigma_2 \end{pmatrix}$

from eq. (22) we obtain

$$R^*(s, x, z) = T(z)\, \Delta\, \{e^{\tau_1 x}, e^{\tau_2 x}\}\, T^{-1}(z)\, R^*(s, 0, z) \qquad (24)$$

Where $T(z) = \begin{pmatrix} \sigma + \frac{\varepsilon(z)}{2\sigma_1} & \sigma\ \frac{\varepsilon(z)}{2\sigma_1} \\ 1 & \cdot 1 \end{pmatrix}$, $r^{-1}(z) = \frac{\sigma_1}{\varepsilon(z)} \begin{pmatrix} 1 & \frac{\varepsilon(z)}{2\sigma_1} - \sigma \\ = 1 & \frac{\varepsilon(z)}{2\sigma_1} + \sigma \end{pmatrix}$

so, from eq. (23), (24) we get

$$R^*(s, 0, z) = T(z)\, \Delta \left\{ \frac{z}{z-g^*(-\tau_1)}, \frac{z}{z-g^*(-\tau_2)} \right\} T^{-1}(z) \left(e_1 - aP^*_0(s) \right) \qquad (25)$$

$$R^*_2(s, 0, z) = \frac{\sigma_1{}^z\ (1,1)}{\varepsilon(z)} \begin{pmatrix} \frac{1}{z-g^*(-\tau_1)} \left\{ 1-a[P^*_{01}(s) + (\frac{\varepsilon(z)}{2\sigma_1} - \sigma)\, P^*_{02}(s)] \right\} \\ \frac{1}{z-g^*(-\tau_1)} \left\{ 1-a[P^*_{01}(s) + (\frac{\varepsilon(z)}{2\sigma_1} + \sigma)\, P^*_{02}(s)] \right\} \end{pmatrix} \qquad (26)$$

By Rouche Theorem $z = g^*(-\tau_1)$, during $|z| \le 1$, it has inside unit circle, write as z_1, $i = 1, 2$; and $R^*(s, 0, z)$. $|z| \le 1$, from eq. (26) we know $P^*_0(s)$ should fit for equation.

$$\begin{pmatrix} 1 & \frac{\varepsilon(z)}{2\sigma_1} - \sigma \\ -1 & \frac{\varepsilon(z)}{2\sigma_1} + \sigma \end{pmatrix} \cdot P_0^*(s) = \begin{pmatrix} \frac{1}{s + \lambda - \lambda z_1} \\ \frac{1}{s + \lambda - \lambda z_1} \end{pmatrix} \qquad (27)$$

we get

$$P^*_0(s) = \frac{1}{\varepsilon(z_1)+\varepsilon(z_2)} \begin{pmatrix} \sigma_2 - \sigma_1 + \varepsilon(z_2) & \sigma_2 - \sigma_1 - \varepsilon(z_2) \\ 2\sigma_1 & 2\sigma_2 \end{pmatrix} \begin{pmatrix} \frac{1}{s+\lambda-\lambda z_1} \\ \frac{1}{s+\lambda-\lambda z_2} \end{pmatrix} \quad (28)$$

$$P^*_1(s,x,z) = \bar{G}(x) \begin{pmatrix} \sigma + \frac{\varepsilon(z)}{2\sigma_1} & e\tau_1^{x} \; e\tau_2^{x} \end{pmatrix} \tau^{-1}(z)\, R^*(s,o,z)\, e^{-ay} \quad (29)$$

4. System Characteristics

LEMMA 1:- $\lim_{s \to 0} s \left[P^*_{01}(s) + P^*_{02}(s)\right] = 1 - p$,

where $p = \frac{\lambda}{\mu}\left(1 + \frac{\sigma_1 \sigma_2}{\beta(\sigma_1 + \sigma_2)}\right)$

Proof:- We notice $\lim_{s \to 0} z_1 = 1$, $\lim_{s \to \sigma} z_2 \neq 1$, $\lim_{s \to 0} \varepsilon(z_1) = \sigma_1 + \sigma_1$

$\lim_{s \to 0} \tau_1 = 0$, $\lim_{s \to 0} h^{*}(s) = -\frac{1}{\beta}$, $\lim_{s \to 0} g^{*}(s) = -\frac{1}{\mu}$, use z_1, is fit for $z_1 = g^{*}(-\tau_1)$ we obtain.

$\lim_{s \to 0} g \frac{dz_1}{ds} = \frac{p}{\lambda(p-1)}$, $\lim_{s \to 0} \frac{1}{1-\lambda z_1} = 1 - e$

Then using eq. (28) we can easily prove.

Theorem 1:- if $p < 1$, the idle probability of the server is positive and consequently the system can reach stable equilibrium.

Proof: If the system can reach stable equilibrium then

$\lim_{t \to \infty} \left[P_{01}(t) + P_{02}(t)\right] = \lim_{s \to 0} \left[s\, P^{*}_{01}(s) + P^{*}_{02}(s)\right]$ existing and > 0

And from lemma (1) we know that

$\lim_{s \to \infty} s \left[P^*_{01}(s) + P^*_{02}(s)\right] = 1 - p$, so $p < 1$

If $p < 1$ from lemma (1) then

$\lim_{t \to \infty} \left[P_{01}(t) + P_{02}(t)\right] = 1 - p > 0$ that is the system can reach stable equilibrium.

Theorem 2:- The laplace transform of the systems renewal density is

$$m^*(s) = \frac{s+\lambda}{\varepsilon(z_1)+\varepsilon(z_2)}\left\{\frac{\sigma_2-\sigma_1+\varepsilon(z_2)}{s+\lambda-\lambda z_1} - \frac{\sigma_2-\sigma_1+\varepsilon(z_1)}{s+\lambda-\lambda z_2}\right\} - 1 \quad (30)$$

$$= \frac{\lambda}{\varepsilon(z_1)+\varepsilon(z_2)}\left(\frac{\sigma_2-\sigma_1+\varepsilon(z_2)}{s+\lambda-\lambda z_1} z_1 - \frac{\sigma_2-\sigma_1+\varepsilon(z_1)}{s+\lambda-\lambda z_2}\right) \quad (31)$$

COROLLARY 1:- The Laplace transform of the systems renewal time distribution is

$$m = \lim_{s\to 0} sm^*(s) = \frac{\lambda\left(\sigma_1-\sigma_2+\varepsilon(\bar{z}_2)\right)}{(\sigma_1-\sigma_2+\varepsilon(\bar{z}_2))}(1-p), \quad \bar{z}_2 = \lim_{s\to 0} s_2$$

Proof: we notice system renewal frequency

$$m(t) = \int_0^\infty \mu(x) P_{11}(t,x)\,dx$$

According to Ross[7] and Theorem (2) the proof follows immediately.

LEMMA 2:- The Laplace-Stieltjes transform of the systems renewal distribution is

$$f(s) = \frac{m^*(s)}{1-m^*(s)} = \frac{\lambda}{s+\lambda} \quad \frac{v_1 z_1 - v_2 z_2}{v_1 - v_2}$$

where $v_1 = \frac{\sigma_2-\sigma_1+\varepsilon(z_2)}{s+\lambda-\lambda z_1}$ $\quad V_2 = \frac{\sigma_2-\sigma_1+\varepsilon(z_2)}{s+\lambda-\lambda z_2}$

5. The Reliability Indices of the Server

Theorem 3:- The Laplace transform of the server availability is

$$A^*(s) = P^*_{01}(s) + P^*_{02}(s) + \int_0^\infty P^*_{1}(s,x,1)\,dx + \int_0^\infty P^*_{2}(s,x,1)\,dx$$

$$= (1,1)\left[P^*_0(s) + T(1)\,\Delta\{-\tau_1^{-1}, -\tau_2^{-1}\}\, T^{-1}(1)\,(\hat{e}_1 - sP^*_0(s)\right] \quad (32)$$

Proof: The proof is obvious.

COROLLARY:- The limiting availability of the server is

$$A = \lim_{s \to 0} sA^*(s) = 1 - \frac{\lambda\sigma_1\sigma_2}{\beta\mu(\sigma_1 + \sigma_2)} \tag{33}$$

Theorem 4:- The Laplace transform of the server failure frequency is

$$W_r^*(s) = \sigma_2 \int_0^\infty P^*_2(s, x, 1)\, dx$$

$$= \sigma_2 [-\tau_1^{-1}, -\tau_2^{-1}]\, r^{-1}(1) \left(e_1 - sP^*_0(s)\right) \tag{34}$$

Proof:- Using the general formula of Ref. [6] we can easily prove.
COROLLARY:- The limiting failure frequency of the server is

$$W_r(s) = \lim_{s \to 0} sW^*_r(s) + \frac{\lambda\sigma_1\sigma_2}{\mu(\sigma_1 + \sigma_2)} \tag{35}$$

Proof:- We notice that

$$W_r(t) = \sigma_2 \sum_{n=1}^{\infty} \int_0^\infty P_{n2}(t, x)dx = \sigma_2 \int_0^\infty P_2(t, x)dx,$$

then the prove is obvious.
In order to find the first failure time distribution of the server, we regard (n. 3, x, y) (n = 1, 2, ...;) as an absorbing state see fit. 1. In this case, the set of equations which the state probabilities must satisfy.

$$\left(\frac{\partial}{\partial t} + \lambda\right) Q_{01}(t) = \int_0^\infty \mu(x)\, Q_{11}(t, x)dx, \; i = 1, 2; \tag{36}$$

where $Q_{01}(t) = \{P\; s(t) = 0, I(t) = 1\}$, $\quad i = 1, 2,$

$Q_{ni}(t, x)\, dx = P\{S(t)=n,\; I(t) = 1,\; x \le x(t) < x + dx\}$,
$i = 1, 2;\; n = 1, 2..;\; 0 \le x < \infty$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \sigma_1 + \mu(x)\right) Q_{n1}(t, x) = \lambda Q_{n-11}(t, x),\; n = 1,2\ldots; \tag{37}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \sigma_2 + \mu(x)\right) Q_{21}(t, x) = \lambda Q_{n-12}(t, x) + \sigma_1 Q_{n1}(t,x) \tag{38}$$

$n = 1, 2 \ldots;$
we write all $Q_{01}(t, x) = 0$, $\quad i = 1, 2;$

$$Q_{n1}(t, 0) = \lambda\, \mu(t)\, Q_{01}(t)^{\delta}{}_{1n} + \int_0^{\infty} \mu(x)\, Q_{n-11}(t, x)\, dx, \; i = 1, 2 \ldots; \quad (39)$$

The initial conditions are $Q_{01}(0) = 1$ and others are zero.
Taking Laplace transform about t of eq. (36) and (39) and then z-transform dividing both side by $\bar{G}(x)$, we get.

$$\frac{\partial}{\partial x} q^*(s, x, z) = \begin{pmatrix} -b_1 & 0 \\ -\sigma_1 & -b_2 \end{pmatrix} q^*(s, x, z)$$

$$zq^*(s, x, z) = \int_0^{\infty} g^*(x)\, q^*(s, x, z)\, dx = z\,[e_1 - Q^*_0(s)]$$

Solving the set of equations, we obtain

$$q^*(s, x, z) = T_0\, \Delta\, \{e^{-b_1 z}, e^{-b_2 x}\}\, T_0^{-1}\, q^*(s, 0, z) \quad (40)$$

where

$$T_0 = \begin{pmatrix} 1 & 0, \\ \frac{\sigma_1}{\sigma_2 - \sigma_1} & 0_1 \end{pmatrix} \quad T_0^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{\sigma_1}{\sigma_2 - \sigma_1} & 1 \end{pmatrix}$$

$$q^*(s, 0, z) = zT_0\Delta \left\{ \frac{1}{z - g^*(b_1)}, \frac{1}{z - g^*(b_2)} \right\} T^{-1}_0\, [e_1 - SQ^*_0(s)]$$

$$Q^*_0(s) = \frac{1}{s+\lambda+\lambda u_1} \left(1, \; \frac{\lambda\,\sigma_1}{\sigma_{1s} - \sigma_2} \; \frac{u_2 - u_1}{s+\lambda+\lambda u_1} \right) \quad (41)$$

u_1 is the equation $z = g^*(s+\lambda+\sigma_1 - \lambda z)$ be the unique root, when $|z| \le 1$, $i = 1, 2, \ldots\ldots;$

Theorem 5: The laplace transform of the rebiality of service station

$$R^*(s) = (1,1) \left(Q^*_0(s) = T_0\, \Delta \left\{ \frac{1}{s + \sigma_1}, \frac{1}{s + \sigma_2} \right\} T_0^{-1}\, [e_1 - sQ^*_0(s)] \right) \quad (42)$$

Proof: we know $R^*(s) = (1,1)\, [Q^*_0(s) + \int_0^{\infty} \bar{G}(x)\, q^*(s, x, 1) dx]$, then we will

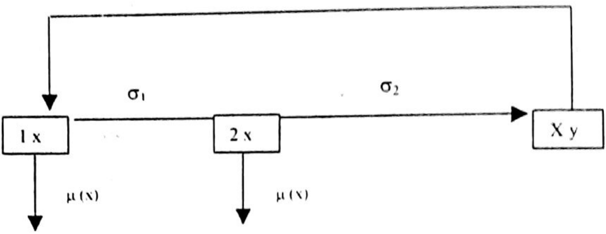
Get proof

COROLLARY:- The mean time to the first failure (MTTFF) of the server is

$$MTTFF = \frac{1}{\lambda(1-\bar{u}_1)}\left(1+\frac{\sigma_1}{\sigma_1-\sigma_2}\;\frac{\bar{u}_2-\bar{u}_1}{1-\bar{u}_2}\right)+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}$$

where $\bar{u}_i = \lim_{s \to 0} u_i$, $i = 1,2$

6. Distribution of the time of a Customer Spent on the Server.



Because the time customer spend on server has no relation with queue length n. So only need to consider about following states use $S_1(t) = 1$, where system run in the stable equilibrium at time t service life is in Now here $i = 1, 2$.

Where $X_i(t)$ indicate the service time of the customer spent $0 \le x_1(t) < \infty$ and $S_1(t) = 3$, indicate the service station is being fixed at time t. Lead in the replenish $Y_1(t)$ indicate the fixing time service station had used $0 \le Y_1(t) < \infty$ easy to know process $\{S_1(t), X_1(t), Y_1(t)\}$ is state dimensions vector Markov Process. The state transition is shown as in the figure. They fit for following calculus equations.

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\sigma_1+\mu(x)\right] r_1(t, x) = \int_0^{\infty} \beta(y)\, r_3(t, x, y)\, dy \qquad (44)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\sigma_2+\mu(x)\right] r_2(t, x) = \sigma_1 r_1(t, x) \qquad (45)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta(x)\right] r_3(t, x, y) = 0 \qquad (46)$$

Boundary conditions are

$$r_i(t, 0) = \frac{1}{\sigma_i} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} s(t), \qquad i = 1, 2; \qquad (47)$$

$$r_3(t, x, 0) = \sigma_2 r_2(t, x) \qquad (48)$$

Here initial conditions are

$$r_i(0, x) = \frac{1}{\sigma_i} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} s(t), \qquad i = 1, 2; \; r_3(0, x, y) = 0 \qquad (49)$$

Initial conditions (49) is according to the following lemmas.

LEMMA 2:- In stable equilibrium, the probability that service to a customer starts from the ith phase of the server life, given that server is busy, is

$$p_i = \frac{1}{\sigma_i} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \qquad i = 1, 2;$$

It has no relation to the life phase of the server.

Proof:- The conditional probability that a customer is in the ith phase of the server life, given the server is busy, can be written as

$$\lim_{s \to 0} \sum_{i=1}^{2} s \int_0^\infty P^*_i(s, x, 1)\, dx = \sum_{i=1}^{2} \frac{\lambda \sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sigma_i}$$

$$\text{or } \lim_{s \to 0} s \int_0^\infty P^*_i(s, x, 1)\, dx = \frac{\lambda \sigma_1 \sigma_2}{\mu(\sigma_1 + \sigma_2)} \frac{1}{\sigma_i}$$

By the memoryless property of the exponential distribution, the above probability is also the probability that service to the customer starts from the ith phase of the server life. Given the server is busy.

So we get the initial condition (49) use eq. (44) – (49) obtain

$$r^*(s, x) = \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} T(1)\, \Delta \left\{ \bar{G}(x) e^{\tau_1 x}, \; \bar{G}(x) e^{\tau_2 x} \right\} T^{-1}(1) \begin{pmatrix} \frac{1}{\sigma_1} \\ \frac{1}{\sigma_2} \end{pmatrix}$$

$$r^*_3(s, x, y) = \sigma_2 \bar{H}(y)\, r^*_2(S, x)\, e^{-ay}$$

Further we get

$$\int_0^\infty r^*(s,x)\,dx = \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\; T(1)\,\Delta\,\left\{\bar{G}(-\bar{\tau}_1),\; \bar{G}(-\bar{\tau}_2)\right\} T^{-1}(1)\begin{pmatrix}\frac{1}{\sigma_1}\\ \frac{1}{\sigma_2}\end{pmatrix} \quad (50)$$

$$\int_0^\infty\int_0^\infty r^*(s,x)\,dx\,dy = \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\; \bar{H}(s)\,\left\{\bar{G}(-\bar{\tau}_1),\; G(-\tau_2)\right\} \bar{r}^{-1}(1)\begin{pmatrix}\frac{1}{\sigma_1}\\ \frac{1}{\sigma_2}\end{pmatrix} \quad (51)$$

Proof can easly get from eq. (50), (51).

Theorem 6:- If one can write the time distribution which customer spend on the service station as $G_0(t)$ that's $G_0(t) = P\{x \le t\}$, then

$$G_0(t) = \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\left\{C_1\bar{G}(-\bar{\tau}_1),\; \bar{C}_2\,\bar{G}(-\tau_2)\right\} T^{-1}(1)\begin{pmatrix}\frac{1}{\sigma_1}\\ \frac{1}{\sigma_2}\end{pmatrix} \quad (52)$$

where $C_1 = \dfrac{\sigma_1+\sigma_2+\sqrt{(\sigma_1-\sigma_2)^2+4\sigma_1\sigma_2 h^*(s)}}{2\sigma_1} + \sigma_1\;\bar{H}(s)$

$$C_1 = \frac{\sigma_1+\sigma_2+\sqrt{(\sigma_1-\sigma_2)^2+4\sigma_1\sigma_2 h^*(s)}}{2\sigma_2} + \sigma_2\;\bar{H}(s)$$

COROLLARY:- The mean time that a customer spent on the server is

$$E[x] = \frac{1}{\mu}\quad 1+\left(\frac{\sigma_1+\sigma_2}{\beta(\sigma_1+\sigma_2)}\right) = \frac{p}{\lambda} \quad (53)$$

If we change the initial and boundary conditions of eq. (47 – (49) as

$r_1(t,0) = S(t)\; r_2(t,0) = 0,\; r_3(t,x,0) = \sigma_2 r_2(t,x),$

$r_1(0,x) = S(x).\qquad r_2(0,x) = r_3(0,x,y) = 0,$

$r_1(t,0) = 0,\; r_2(t,0) = \delta(t).\; r_3(t,x,0) = \sigma_2 r_2(t,x)$

$i_1(0,x) = 0,\; r_2(0,x) = \delta(x).\; r_3(0,x,y) = 0$

We can get following results.

Theorem 7:- When system run to time t and service life is at 1, 2 phase the Laplace transform of distribution $\bar{G}^*_1(s)$, $\bar{G}^*_2(s)$, which the time from customer begin to accept service to service finish is,

$$\bar{G}^*_1(s) = \{c_1\;\bar{G}(-\bar{\tau}_1),\; c_2\;\bar{G}(-\bar{\tau}_2)\}\; T^{-1}(1)\, e_1\quad i = 1, 2; \quad (54)$$

The mean time spent on the service station is

$$E[X_1] = \frac{p}{\lambda} - \frac{\sigma_1 \sigma_2}{\beta(\sigma_1 + \sigma_2)} \bar{G}(\sigma_1 + \sigma_2) \quad (55)$$

$$E[X_2] = \frac{p}{\lambda} + \frac{\sigma_1 \sigma_2}{\beta(\sigma_1 + \sigma_2)} \bar{G}(\sigma_1 + \sigma_2) \quad (56)$$

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