



Proper curvature collineations of plane symmetric static spacetime in $f(R)$ theory of gravity

M. RAMZAN⁺⁺, A. NAZIR, M. R. MUFTI*

Department of Mathematics, The Islamia University of Bahawalpur, 63100, Pakistan

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Abstract: The purpose of this paper is to study the proper curvature collineations of plane symmetric static spacetime in $f(R)$ theory of gravity. We have taken the metrics of both constant and non-constant curvature and formulated the curvature vector fields by using vanishing Lie derivative and direct integration technique. Here, the solutions of the Einstein's field equations (EFEs) for both cases involve inverse curvature terms which may cause an increase in gravity that justifies the accelerated expansion of the universe. It turns out that curvature collineations become Killing vector fields and therefore, proper curvature collineations do not exist in both the cases.

Keywords: Proper curvature collineations; Lie derivative, Direct integration technique, $f(R)$ theory of gravity.

1. INTRODUCTION

The spacetime admitting the three-parameter group of motions of the Euclidean plane is said to possess a plane symmetry and is known as plane-symmetric spacetime. Many properties of these spacetimes are similar to those of spherically symmetric spacetime. Many researchers have studied plane-symmetric spacetime in different ways. Taub (Taub, 1951), Bondi (Bondi, 1957), Bondi and Pirani-Robinson (Bondi, *et al.*, 1959) investigated plane-wave solutions.

Nowadays, $f(R)$ theory of gravity has become an active field of research. Researchers believe that the modification of Einstein's theory with some inverse curvature terms may cause an increase in gravity that justifies the accelerated expansion (Capozziello, *et al.*, 2003) and (Carroll, *et al.*, 2004).

Recent literature (Sotiriou, 2006), (Amendola, *et al.*, 2007), (Sharif and Shamir, 2010), (Shamir and Raza, 2015), (Felice and Tsujikawa, 2010), (Sotiriou and Faraoni, 2010), (Clifton, *et al.*, 2012), (Nojiri and Odintsov, 2011), (Bamba, *et al.*, 2014), (Bamba, *et al.*, 2010), (Bamba, *et al.*, 2011), (Capozziello and Vignolo, 2011), (Capozziello, *et al.*, 2011) and (Elizalde, *et al.*, 2010) shows deep interest in exploring different problems in $f(R)$ theory of gravity. Spherically symmetric solutions are the most widely explored exact solutions in $f(R)$ gravity. Spherically symmetric vacuum solutions were studied in (Multamaki and Vilja, 2006) and it was found that the set of the field equations in $f(R)$ theory of gravity gave the Schwarzschild de-Sitter metric. Nonvacuum plane symmetric static solutions of Einstein's field equations in the metric of $f(R)$ theory of gravity with the

condition of constant and non-constant curvature was explored in (Shamir, 2016). These solutions provide Taub's universe with a singularity at $x = 0$, which shows the presence of Black Hole. It can be noticed that with constant curvature assumption, the solution of EFEs becomes similar to Taub's metric (Bedran, *et al.*, 1997).

Here, we are interested to find the curvature collineations (CCs) or curvature vector fields of plane symmetric static spacetime in $f(R)$ theory of gravity with both constant and non-constant curvature assumptions and then we can compare it with curvature vector fields in general relativity (GR). A vector field X is called a curvature collineation (CC) if it satisfies the relation (Katzin, *et al.*, 1969):

$$L_X R^a_{bcd} = 0, \quad (1.1)$$

Or equivalently,

$$R^a_{bcd;e} X^e + R^a_{ecd} X^e_{;b} + R^a_{bed} X^e_{;c} + R^a_{bce} X^e_{;d} - R^e_{bcd} X^a_{;e} = 0.$$

Where L_X denotes the Lie derivative of Riemann curvature tensor R^a_{bcd} along the vector field X and $(;)$ denotes the covariant derivative.

The vector field X is said to be proper curvature collineation if it is not affine (Hall and da Costa, 1991). One can expand the above equation in a set of 22 coupled CC equations (Shabbir, *et al.*, 2003) given below

⁺⁺Corresponding author: M. RAMZAN email: muhammad.ramzan@iub.edu.pk

* Department of Computer Science, COMSATS, Institute of Information Technology, Vehari, Pakistan

$$\begin{aligned}
R_{1a1}^a X_{,0}^1 + R_{0b0}^b X_{,1}^0 &= 0, (a, b) = (0, 1), (2, 2), (3, 3), \\
R_{2a2}^a X_{,0}^2 + R_{0b0}^b X_{,2}^0 &= 0, (a, b) = (0, 2), (1, 1), (3, 3), \\
R_{3a3}^a X_{,0}^3 + R_{0b0}^b X_{,3}^0 &= 0, (a, b) = (0, 3), (1, 1), (2, 2), \\
R_{1a1,e}^a X^e + 2R_{1a1}^a X_{,1}^1 &= 0, a = 0, 2, 3 \text{ and } e = 0, 1, 2, 3, \\
R_{1a1}^a X_{,2}^1 + R_{2b2}^b X_{,1}^2 &= 0, (a, b) = (0, 0), (2, 1), (3, 3), \\
R_{1a1}^a X_{,3}^1 + R_{3b3}^b X_{,1}^3 &= 0, (a, b) = (0, 0), (3, 1), (2, 2), \\
R_{2a2}^a X_{,3}^2 + R_{3b3}^b X_{,2}^3 &= 0, (a, b) = (0, 0), (1, 1), (3, 2), \\
R_{2a2,e}^a X^e + 2R_{2a2}^a X_{,2}^2 &= 0, a = 0, 1, 3 \text{ and } e = 0, 1, 2, 3, \\
R_{3a3,e}^a X^e + 2R_{3a3}^a X_{,3}^3 &= 0, a = 0, 1, 2 \text{ and } e = 0, 1, 2, 3, \\
R_{0a0,e}^a X^e + 2R_{0a0}^a X_{,0}^0 &= 0, a = 1, 2, 3 \text{ and } e = 0, 1, 2, 3, \\
(R_{0a0}^0 - R_{aba}^b) X_{,b}^0 &= 0, (a, b) = (1, 3), (2, 3), (1, 2), \\
(3, 2), (2, 1), (3, 1), \\
(R_{0a0}^0 - R_{aba}^b) X_{,0}^b &= 0, (a, b) = (1, 3), (2, 3), (1, 2), \\
(3, 2), (2, 1), (3, 1), \\
(R_{aba}^b - R_{aca}^c) X_{,c}^b &= 0, (a, b, c) = (0, 1, 2), (3, 1, 2), \\
(0, 1, 3), (2, 1, 3), \\
(R_{aba}^b - R_{aca}^c) X_{,b}^c &= 0, (a, b, c) = (0, 1, 2), (3, 1, 2), \\
(0, 3, 2), (1, 3, 2), \\
(R_{aba}^b - R_{aca}^c) X_{,b}^c &= 0, (a, b, c) = (0, 1, 3), (2, 1, 3), \\
(0, 2, 3), (1, 2, 3).
\end{aligned}$$

here comma (,) denotes the partial derivative. There is no summation over the repeated indices in above equations. The energy-momentum tensor acts as the source of spacetime curvature. It can be studied in the context of Einstein field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}, \quad (1.2)$$

where R_{ab} is the Ricci tensor, R is the Ricci scalar, g_{ab} is the metric tensor and $\kappa = \frac{8\pi G}{c^4}$, is the coupling constant.

The paper is organized as follows. In section 2, brief introduction of $f(R)$ gravity is given. In section 3, we introduce the metric with constant curvature assumption and discuss the curvature vector fields in detail. In section 4, curvature vector fields are discussed of given metric with non-constant curvature condition. The Results are summarized and concluded in the last section.

2. INTRODUCTION TO $f(R)$ GRAVITY

The $f(R)$ theory of gravity is actually a generalization of general relativity. The action for $f(R)$ gravity is given by (Multamaki and Vilja, 2006)

$$S = \int \sqrt{-g} \left[\frac{1}{16\pi G} f(R) + L_m \right] d^4x. \quad (2.1)$$

Here, $f(R)$ is a generic function of Ricci scalar (R) and L_m , is called matter Lagrangian. This action is obtained by replacing R by $f(R)$ in the standard Einstein-Hilbert action. The field equations can be found by varying the action with respect to the metric $g_{\mu\nu}$,

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = \kappa T_{\mu\nu}, \quad (2.2)$$

where $T_{\mu\nu}$ is the standard energy-momentum tensor and

$$F(R) \equiv \frac{df(R)}{dR}, \quad \square \equiv \nabla^\mu \nabla_\mu \quad (2.3)$$

where ∇_μ is the covariant derivative. These modified field equations given in (2.2) are fourth-order partial differential equations in the metric tensor. If we take $f(R) = R$ in (2.2), we can get the field equations in GR.

After contracting (2.2), we get

$$F(R)R - 2f(R) + 3\square F(R) = \kappa T. \quad (2.4)$$

In the case of vacuum

$$F(R)R - 2f(R) + 3\square F(R) = 0. \quad (2.5)$$

The above equation gives the relation between $F(R)$ and $f(R)$, which can be used to simplify the field equations and to evaluate $f(R)$. It can be seen from equation (2.5) that any metric with a constant Ricci scalar $R=R_0$ is a solution of contracted equation (2.5) if the following condition holds:

$$F(R_0)R_0 - 2f(R_0) = 0. \quad (2.6)$$

This condition is known as the ‘‘constant-curvature condition’’. By differentiation (2.5) with respect to R , we get

$$F'(R)R - R'F(R) + 3(\square F(R))' = 0. \quad (2.7)$$

The conditions given in (2.6) and (2.7) were first formulated in 2005 (Cognola, *et al.*, 2005).

3. CURVATURE COLLINEATIONS OF PLANE SYMMETRIC STATIC SPACETIME WITH CONSTANT CURVATURE ASSUMPTION

Here we take the metric of considered spacetime in $f(R)$ gravity with the constant curvature assumption as given in (Shamir, 2016). We find CCs of this metric by solving 22 coupled equations of

Riemann tensor. We use direct integration technique to solve these equations. It is found that the CCs turn in to Killing vector fields in this case.

Metric of plane symmetric static spacetime with constant curvature in f(R) theory of gravity (Shamir, 2016) is

$$ds^2 = -x^{-2/3} dt^2 + dx^2 + x^{4/3} (dy^2 + dz^2). \quad (3.1)$$

Here, nonzero Riemann tensors are

$$R_{101}^0 = -\frac{4}{9x^2}, R_{202}^0 = \frac{2}{9x^{2/3}} = R_{303}^0 = R_{212}^1 = R_{313}^1,$$

$$R_{323}^2 = -\frac{4}{9x^{2/3}}.$$

Here all components of Ricci tensors become zero.

So, in this case scalar curvature is zero ($R=0$).

The 22 coupled curvature equations reduced to be

$$X^1 = 0, \quad (3.1.1)$$

$$X_{,0}^0 = 0, X_{,1}^0 = 0, X_{,2}^0 = 0, X_{,3}^0 = 0, \quad (3.1.2)$$

$$X_{,0}^2 = 0, X_{,1}^2 = 0, X_{,2}^2 = 0, \quad (3.1.3)$$

$$X_{,0}^3 = 0, X_{,1}^3 = 0, X_{,3}^3 = 0, \quad (3.1.4)$$

$$X_{,3}^2 + X_{,2}^3 = 0, \quad (3.1.5)$$

$$2x^2 X_{,0}^2 + X_{,2}^0 = 0, \quad (3.1.6)$$

$$3X_{,0}^3 + \left(-\frac{1}{2x^2} + \frac{2}{x^{8/3}}\right) X_{,3}^0 = 0, \quad (3.1.7)$$

Now, we will solve the above set of equations by using direct integration technique. Here, we can see that the equations (3.1.6) and (3.1.7) are trivially satisfied.

Consider the equations (3.1.1) to (3.1.4), we get

$$\left. \begin{aligned} X^0 &= c_1, \\ X^1 &= 0, \\ X^2 &= D(z), \\ X^3 &= E(y). \end{aligned} \right\} \quad (3.2)$$

Now, consider the equation (3.1.5), differentiating with respect to z and integrating twice we get

$$D(z) = c_2 z + c_3.$$

Using the above value in (3.1.5) and then by solving it, we get

$$E(y) = -c_2 y + c_4.$$

Final solution of system of equations (3.1.1) to (3.1.7) is

$$\left. \begin{aligned} X^0 &= c_1, \\ X^1 &= 0, \\ X^2 &= c_2 z + c_3, \\ X^3 &= -c_2 y + c_4 \end{aligned} \right\} \quad (3.3)$$

Here, C_i 's are the constants of integration. We can see that CCs have become Killing vector fields. Where the Killing vector fields with respect to the metric given in (3.1) are

$$\frac{\partial}{\partial t}, z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

We can see from equation (1.2) that all the components of energy momentum tensor are zero as

$$T_{ab} = 0, \quad a = b = 0, 1, 2, 3.$$

It shows that energy density and pressure is zero.

4. CURVATURE COLLINEATIONS OF PLANE SYMMETRIC STATIC SPACETIME WITHOUT CONSTANT CURVATURE ASSUMPTION

We consider the metric of plane symmetric static spacetime for non-constant curvature that is given in (Shamir, 2016). After finding the CCs, we can see that they become Killing vector fields.

Metric of plane symmetric static spacetime with non-constant curvature in f(R) theory of gravity (Shamir, 2016) is

$$ds^2 = -\frac{1}{x} dt^2 + dx^2 + \frac{1}{x^{11/3}} (dy^2 + dz^2). \quad (4.1)$$

Here, the nonzero components of Riemann and Ricci tensors are

$$R_{101}^0 = -\frac{3}{4x^2}, R_{202}^0 = -\frac{11}{12x^{17/3}} = R_{303}^0,$$

$$R_{212}^1 = -\frac{187}{36x^{17/3}} = R_{313}^1, R_{323}^2 = -\frac{121}{36x^{17/3}}.$$

$$R_{00} = -\frac{31}{12x^3}, R_{11} = -\frac{401}{36x^2},$$

$$R_{22} = -\frac{341}{36x^{17/3}} = R_{33}.$$

Here $R = -\frac{55}{2x^2}$, represents non constant

curvature.

According to these values, the 22 coupled curvature equations take the form

$$X^1 = 0, \quad (4.1.1)$$

$$X_{,0}^0 = 0, X_{,1}^0 = 0, X_{,2}^0 = 0, X_{,3}^0 = 0, \quad (4.1.2)$$

$$X_{,0}^2 = 0, X_{,1}^2 = 0, X_{,2}^2 = 0, \quad (4.1.3)$$

$$X_{,0}^3 = 0, X_{,1}^3 = 0, X_{,3}^3 = 0, \quad (4.1.4)$$

$$X_{,3}^2 + X_{,2}^3 = 0, \quad (4.1.5)$$

$$199X_{,0}^2 + 31X_{,2}^0 = 0, \quad (4.1.6)$$

$$199X_{,0}^3 + 31X_{,3}^0 = 0. \quad (4.1.7)$$

Now, we will solve the above set of equations by using direct integration technique. Here, we can see

that the equation (4.1.6) and (4.1.7) are trivially satisfied.

Consider the equations (4.1.1) to (4.1.4), we get

$$\left. \begin{aligned} X^0 &= c_1, \\ X^1 &= 0, \\ X^2 &= G(z), \\ X^3 &= H(y). \end{aligned} \right\} \quad (4.2)$$

Now, consider the equation (4.1.5), differentiating with respect to z and integrating twice we get

$$G(z) = c_6 z + c_7.$$

Using the above value in (4.1.5) and then by solving it, we get

$$H(y) = -c_6 y + c_8.$$

Final solution of system of equations (4.1.1) to (4.1.7) which shows the curvature vector fields are

$$\left. \begin{aligned} X^0 &= c_5, \\ X^1 &= 0, \\ X^2 &= c_6 z + c_7, \\ X^3 &= -c_6 y + c_8 \end{aligned} \right\} \quad (4.3)$$

The above result shows that the curvature vector fields turn to Killing vector fields. Following are the Killing vector fields with respect to the metric given in (4.1)

$$\frac{\partial}{\partial t}, z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

Here by using (1.2), we get the components of energy momentum tensors as

$$T_{00} = \frac{55}{4x^3}, T_{11} = -\frac{187}{36x^2}, T_{22} = -\frac{247}{36x^{17/3}},$$

$$T_{33} = -\frac{247}{36x^{17/3}}.$$

which represent that energy density of considered spacetime can not be zero but can be maximized to infinity.

5. SUMMARY AND DISCUSSION

In this paper, we have taken the plane symmetric static solutions of Einstein's field equation in $f(R)$ theory of gravity with both constant and non-constant curvature assumption that are given in (Shamir, 2016). We find the CCs by solving 22 coupled curvature equations in both cases. We see that in both cases CCs become Killing vector fields. The energy momentum tensors in the case of constant curvature become zero but in the case of non-constant curvature, the components of energy momentum tensor are non-zero and $T = \infty$ at the singular point means that energy density is maximum. In the non-constant curvature case the

rank of 6x6 Riemann matrix is six which shows that no proper curvature collineations are admitted (Hall and da Costa, 1991) in the considered plane spacetime. In the light of $f(R)$ theory of gravity the CCs are the isometries for the given spacetime. The existence of Killing vector fields or Isometry in the modified theory of gravity imply the conservation laws which show the physical importance of the theory.

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