*Sindh Univ. Res. Jour. (Sci. Ser.) Vol.49(2) 283-288 (2017)*

**SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)**

# Exact Solution of Tank Drainage for Newtonian Fluid with Slip Condition

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Received 23rd January 2016 and Revised 19th February 2017

**Abstract:** This paper explores the problem of tank drainage of an unsteady, incompressible, isothermal Newtonian fluid with slip condition. The exact result is obtained from governing continuity and momentum equation's focus to proper preconditions. The Newtonian solution without slip condition is retrieved from this proposed model on substitution β=0.Declaration on behalf of velocity profile, volume flux, average velocity, shear stress over the pipe, relationship how does the time vary with length and time required for complete drainage are obtained. Effects of various emerging parameters on velocity profile $v_z$and depth *H(t)* are presented graphically.

**Keywords**: Tank drainage, Newtonian fluid, Slip condition, Exact solution.

## 1. INTRODUCTION

The quandary of flow with slip-stream is very critical in this time of present day science, development and infeasible running industrialization. The wonder of slip-stream regime has pulled in the consideration of an expansive number of researchers because of its far reaching application. The constituent part nearby a strong surface no lengthier considers the velocity of the surface in various certifiable applications. At the surface of a particle causes a determinate tangential velocity; in this situation it skims on the particle. This current organism is named as slip flow regime and their force can't be overlooked. Slippage of fluid marvel placed at stable limits show up in numerous applications, for example, small scale channels or Nano directs and in utilization wherever a slight layer of "light oils" is connected toward the moving-plates on the other hand at the same time surface remains covered with unique covering, for example, thick monolayer of hydrophobic octa-decyltric hlorosilane (Derek, *et al.*,2002) i.e. grease of mechanical gadget wherever a dainty layer of oil is connected to the surface slipping more than each other or when the surfaces are covered with exceptional covering to minimize the grinding between them.In the historical backdrop of liquid course through channels, Navier examined a limit state of liquid slip at strong surface such that $v_z = -\beta S_{rz}\big|_{r=R}$ , where $\beta$ is the slip coefficient, $S_{rz}$ is part of extra stress tensor and $v_z$ is the velocity along *z-axis*. For a situation $\beta = 0$ lists that there is no slip at the limit.

The drainage of a fluid through pipe of a tank under the action of gravity is an old, how ever complicated problem. The tank may be drained by an attach pipe or may be drained throughevenhanded hole "orifice situation". The pipe possibly could be horizental or vertical or may contain a complete piping system with horizental extension and vertical drop with fittings and valve, etc. Usually tank has a shape of cylinderical contain a vertical wallhoweverbottom maybe conical hemisherical or by flat or might be additional shape. There is some time sintrest in draining the tank should be totally dry in which situation the bottom shape needs to be accounted for and occasionally not.

Classifications of gravity draining fluid's are used extensively throughout industries, a small number ofsuch classifications are: draining condensate into storage, water distribution, waste water management and dams, retrieval of chemicals from tank farm. The generated model will accurately represent tank draining behavior for all tanks with a similar setup. End effects, accuracy of time measurement, accuracy of height measurements and friction losses will be taken into consideration (Joe and Macklin, 2005)

An outstanding evaluation of exact solutions of the "Navier-Stokes equation" has been given by (Wang, 1991). In this manuscript, we studied tank drainage problem of Newtonian fluid with slip condition. Exact solutions of the consequential differential equations subject to boundary conditions, are obtained. For the slip parameter $\beta = 0$ , we retrieve velocity profile for linearly viscous case. Also relationships for velocity-profile, flow rate, average-velocity, shear stress on pipe, depth of fluid in the tank and time required for complete drainage are calculated.

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This paper is organized by means of follows: Section number 2 provides basic equation’s for the Navier-Stokes equation. Section number 3 provides formulation and solution of the problem. Section number 4 deals with volume flux, average velocity, shear stress on the pipe, relationship how does the time vary with length and time required for complete drainage. Results and discussion are given in section number 5, while conclusion is provided in section number 6.

## 2 BASIC EQUATIONS

Essential governing equations for incompressible viscous fluid flow, disregarding thermal effects are:

$$\nabla \cdot \mathbf{V} = 0. \quad (1)$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{b} + \nabla \cdot \mathbf{T}, \quad (2)$$

The symbol $\mathbf{V}$ represent velocity vector, $\rho$ denotesthe constant density, $p$ be the dynamic pressure, $\mathbf{b}$ is the body force and $\mathbf{T}$ the extra stress tensor. The operator $\frac{D}{Dt}$ denotes the material derivative. The extra stress tensor describing a Newtonian fluid is made by:

$$\mathbf{T} = \eta \mathbf{A}_1. \quad (3)$$

Here $\eta$ represent dynamic-viscosity and $\mathbf{A}_1$ be the 1st Rivlin Ericksen tensor,represented as:

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T.$$

The cylindrical component’s of equation of motion (1) and (2) are

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad (4)$$

## 3 TANK DRAINAGE

Consider a cylindrical tank containing an incompressible viscous fluid. The radius of the tank is assumed to be $R_T$ and diameter $d$. The initial depth of the fluid is chosen to be $H_0$. The fluid in the tank is drained down by means of a pipe having radius $R$ and length $L$. Further more letting $H(t)$ be the depth of fluid in the tank at any time $t$.Flow of fluid in the pipe is due to gravity and pressure of the fluid in the tank.

We plane to calculate the velocity profile, pressure profile, flow rate, average velocity, shear stress on the pipe, relationship how does the time vary with length and the time required for complete drainage. Here we use cylindrical coordinates $(r, \theta, z)$ with $r$*-axis* normal to the pipe and $z$*-axis* along the center of the pipe in vertical direction. As the flow is individual in the $z$-direction and the $\theta$ and $r$ components of velocity vector $\mathbf{V}$ are equal to zero,

$$V = [v_r, v_\theta, v_z] = [0, 0, v_z(r,t)], \quad (8)$$

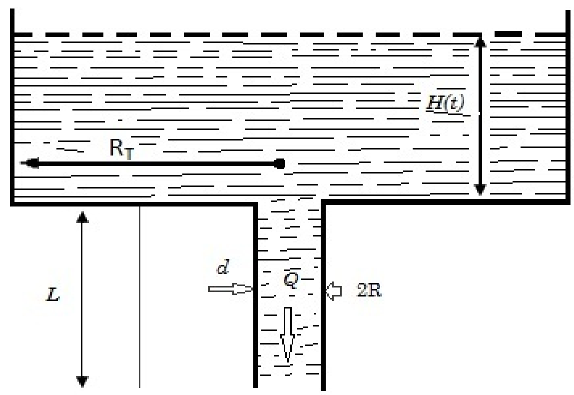



**Fig. 1: Geometry of the tank drainage flow down through pipe**

Using profile (8), the equation of continuity (4) is indistinguishablyfulfilled and the momentum

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right] = -\frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial (r\mathrm{T}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \mathrm{T}_{\theta r}}{\partial \theta} + \frac{\partial \mathrm{T}_{zr}}{\partial z} - \frac{\mathrm{T}_{\theta\theta}}{r}\right] + \rho g_r \quad (5)$$

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial (r^2\mathrm{T}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \mathrm{T}_{\theta\theta}}{\partial \theta} + \frac{\partial \mathrm{T}_{z\theta}}{\partial z}\right] + \rho g_\theta \quad (6)$$

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial (r\mathrm{T}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \mathrm{T}_{\theta z}}{\partial \theta} + \frac{\partial \mathrm{T}_{zz}}{\partial z}\right] + \rho g_z \quad (7)$$

Where , $\mathbf{V} = [v_r(r, \theta, z, t), v_\theta(r, \theta, z, t), v_z(r, \theta, z, t)]$ and $\mathbf{b} = [g_r, g_\theta, g_z]$ gravitational acceleration equation (5-7) diminishes toward

$r$ –*component of momentum*:

$$\frac{\partial p}{\partial r} = 0, \tag{9}$$

$\theta$ –*component of momentum*:

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \tag{10}$$

$z$ –*component of momentum*:

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\eta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g. \tag{11}$$

From equations (9 - 11) we can see that the equation of motion is now quite simple, yielding that the pressure is only function of $z$ and $t$ and the equation to be solved for $v_z(r,t)$ is

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\eta}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \rho g. \tag{12}$$

Equation (12) is a partial differential equation for $p$ and $v_z$. The velocity in the pipe flow remains nearly constant with time due to slow draining such that we may neglect the time derivative $\frac{\partial v_z}{\partial t}$. Also flow in the pipe of radius $R$ is due to both gravity and hydrostatic pressure. The pressures at the pipe entrance and exit are respectively,

$$at \quad z = 0, \quad p = p_1 = \rho g H(t),$$

$$at \quad z = L, \quad p = p_2 = 0,$$

so that

$$\frac{\partial p}{\partial z} = -\frac{\rho g H(t)}{L} \tag{13}$$

The equation of motion (12) thus reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial v_z}{\partial r}\right)\right] = -\frac{\rho g}{\eta}\left[\frac{H(t)}{L} + 1\right]. \tag{14}$$

The associated boundary condition's are

$$at \quad r = 0, \quad \mathrm{T}_{rz} = 0, \tag{15}$$

$$at \quad r = R, \quad v_z = -\beta \mathrm{T}_{rz}\big|_{r=R} \tag{16}$$

Solving equation (14) subject to the boundary conditions (15), we get

$$\frac{\partial v_z}{\partial r} = -\frac{\rho g r}{2\eta}\left(\frac{H(t)}{L} + 1\right). \tag{17}$$

By intigrating equation (17) w.r.to r and by using boudry condition (16), we get

$$v_z = \frac{\rho g}{4\eta}\left(\frac{H(t)}{L} + 1\right)\left(2\beta R\eta + R^2 - r^2\right) \tag{18}$$

Remark: Taking $\beta = 0$ in equation (18), we retrieve the Newtonian solution (Papanastasiou, 1994)

## 4 FLOW RATE, AVERAGE VELOCITY, SHEAR STRESS ON THE PIPE

The "flow rate $Q$" per unit width is specifiedthrough the formula

$$Q = \int_0^R 2\pi r v_z(r,t)\, dr. \tag{19}$$

Using velocity profile (18) in equation (19), one can calculate the flow rate

$$Q = \frac{\rho g \pi}{8\eta}\left(\frac{H(t)}{L} + 1\right)\left(4\beta R^3\eta + R^4\right) \tag{20}$$

We determine the average velocity, $\bar{v}$ by using the formula

$$\bar{V} = \frac{Q}{\pi R^2}. \tag{21}$$

So the average velocity of the fluid flowing down the pipe is

$$\bar{V} = \frac{\rho g}{8\eta}\left(\frac{H(t)}{L} + 1\right)\left(4\beta R\eta + R^2\right) \tag{22}$$

Shear stress on the pipe is given by

$$T_{rz}\,|_{r=R} = -\left(\frac{\rho g R}{2L}\right)\left[H(t) + L\right]. \tag{23}$$

Mass balance over the entire tank is

$$\frac{d}{dt}\left[\pi R_T^2 H(t)\right] = -Q(t). \tag{24}$$

Substituting flow rate from equation (20) into equation (24) and then separating variables on both sides of equation one obtains

$$H(t) = \left(H_0 + L\right)e^{-\frac{\rho g t}{8\eta L R_T^2}\left(4\beta R^3\eta + R^4\right)} - L, \tag{25}$$

and the time required for complete drainage is obtained by taking $H(t) = 0$ in

$$t = \frac{-8\eta L R_T^2}{\rho g(4\beta R^3\eta + R^4)}\ln\left(\frac{H(t) + L}{H_0 + L}\right). \tag{26}$$

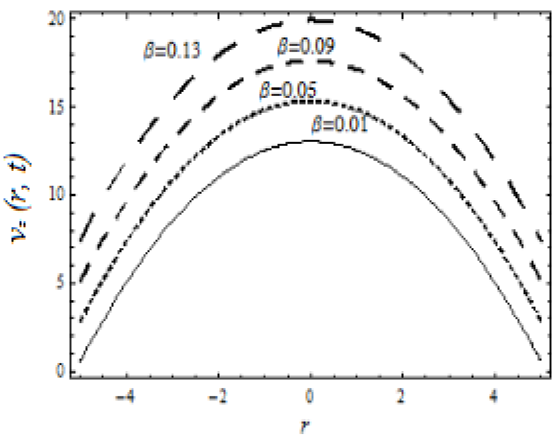
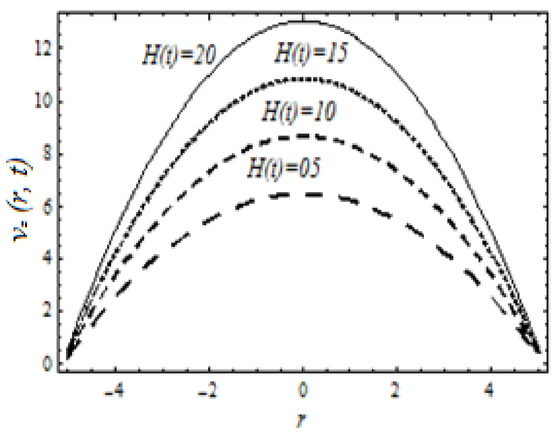


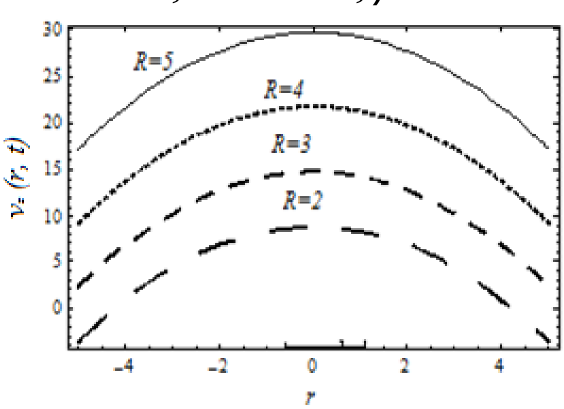

**Fig.2: Effect of $\beta$ on velocity profile, when**

$$\eta = 11.5\, poise, \rho = 0.78\, g / cm^3$$
$$R = 5cm, L = 10cm, H(t) = 20cm.$$




**Fig, 3: Effect of $H(t)$ on velocity profile, when**

$$\eta = 11.5\, poise, \rho = 0.78\, g / cm^3$$
$$R = 5cm, L = 10cm, \beta = 0.01.$$




**Fig, 4: Effect of $R$ on velocity profile, when**

$$\eta = 11.5\, poise, \rho = 0.78\, g / cm^3$$
$$L = 10cm, H(t) = 20cm, \beta = 0.3.$$

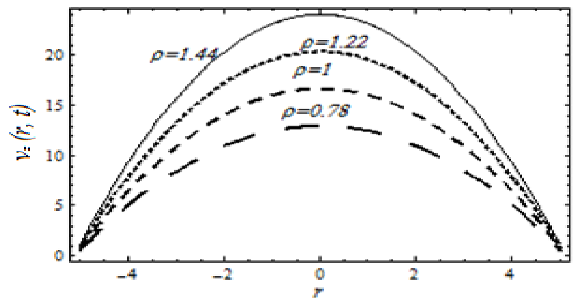
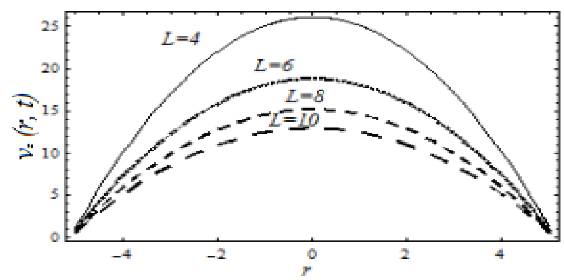


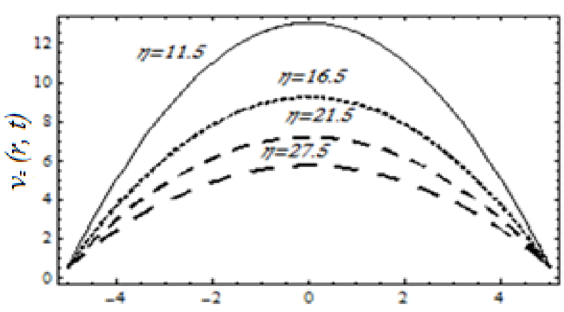

**Fig. 5: Effect of $\rho$ on velocity profile, when**

$$\eta = 11.5\, poise, R = 5cm$$
$$L = 10cm, H(t) = 20cm, \beta = 0.01.$$



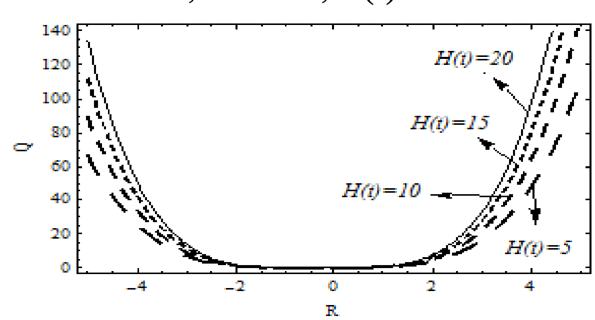

**Fig, 6: Effect of $L$ on velocity profile, when**

$$\eta = 11.5\, poise, \rho = 0.78\, g / cm^3$$
$$R = 5cm, H(t) = 20cm, \beta = 0.01.$$




**Fig,7: Effect of $\eta$ on velocity profile, when**

$$\beta = 0.01, \rho = 0.78\, g / cm^3$$
$$R = 5cm, L = 10cm, H(t) = 20cm.$$




**Fig,8: Effect of $H(t)$ on flow rate, when**

$$\eta = 31.5\, poise, \rho = 0.78\, g / cm^3, L = 10cm, \ \beta = 0.01$$

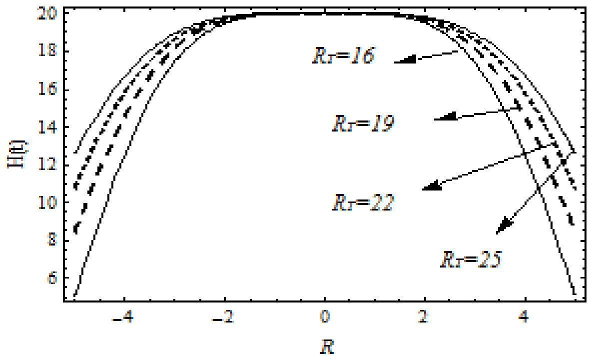
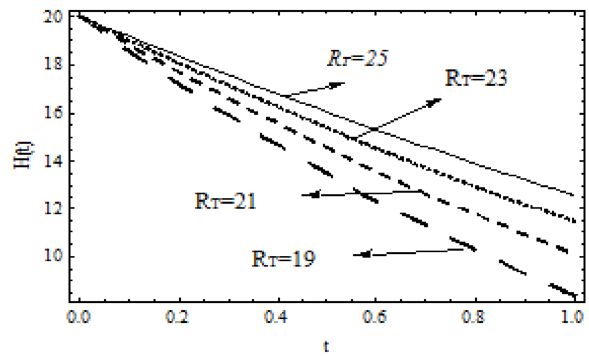



**Fig. 9: Effect of $R_T$ on depth with respect to *R*, when**

$$\eta = 0.6\,poise, \rho = 1.38\,g/cm^3$$
$$t = 1, H_0 = 20cm, L = 10cm, \ \beta = 0.001.$$




**Fig,10: Effect of $R_T$ on depth with respect to *t*, when**

$$\eta = 0.6\,poise, \rho = 1.38\,g/cm^3$$
$$R = 5cm, H_0 = 20cm, L = 10cm, \ \beta = 0.01.$$

## 5 RESULTS AND DISCUSSION

In the above sections we studied tank drainage problem using an incompressible Newtonian fluid with slip condition, exact solutions for the differential equation is obtained. The variation of velocity profile $v_z$, flow rate $Q$ and depth $H(t)$ has been investigated on different parameters. The effects of the slip parameter $\beta$, dynamic viscosity $\eta$, depth $H(t)$, length of pipe $L$, pipe radius $R$ and density $\rho$ on velocity profile are observed through figures (2) - (7)and effect of the depth $H(t)$ on flow rate is shown in figure (8)and effect of the radius of tank $R_T$ on depth $H(t)$ is examined in figure (9) – (10). In figures(2) – (7) it is detected that the magnitude of velocity increases as the increase with slip parameter $\beta$, depth $H(t)$, pipe radius $R$ and density $\rho$ and decreases for the increase of length of pipe $L$ and dynamic viscosity $\eta$ .In figure 8 for the increase $H(t)$ we detected that flow rate increasesand in figures (9) –(10) depth $H(t)$ with respect to pipe radius $R$ as well as for time $t$ are plotted, in both cases depth $H(t)$ increases with increase of radius of tank $R_T$ .

## 6 CONCLUSIONS

Considering equation for unsteady, incompressible, isothermal tank drainage flow for the Newtonian fluid with slip condition. We have obtained exact solutions for “velocity profile, flow rate, average velocity and shear stress on the pipe”. Here itis noted that for the slip parameter $\beta = 0$, solution (18) reduces to the Newtonian solution without slip condition (Papanastasiou, 1994). A relationship (26), how does the time vary with length is derived. It is notedthat as the fluid is becoming thicker, velocity of the fluid decreases.

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