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Analytical Results of Viscoelastic Fluid Flow through Porous Media in a Circular Pipe Employing Oldroyd–B Constitutive Model

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Abstract: The basic purpose of this research paper is to investigate the solution of velocity of viscoelastic fluid flow with Porous media in a circular pipe to use the Lei- group technique to the system of PDE's comprising the continuity, momentum and constitutive; equations, under appropriate initial and boundary conditions. The associated problem is organized by using the Darcy-Brinkman model connected with Oldroyd–B constitutive model. The investigation is arranged through analytical solutions of the PDE's system related with initial and boundary value problem. The analytical solution is found under the symmetries for the system through Lie group method. The system of PDE's is reduced in to ODE's system applying suitable boundary conditions, so that exact solutions can be obtained.

Keywords: Solution of Viscoelastic fluid flow with Porous media in circular pipe, Darcy-Brinkman model, Oldroyd–B constitutive model, Lie Group Technique.

1. INTRODUCTION

Mechanics is the basic of sciences. It tends to provide better understanding of the physical world along with developing various skills and strange fields of interests such as fluid dynamics. In the fluid dynamics, Newtonian and non-Newtonian fluids are meaningfully worried enormous to interest in the literature. The results of Newtonian and Non–Newtonian fluid flow in problems classically depend on calculation of different properties of the flow of fluid. Flows of Newtonian and non-Newtonian fluids related with some essential investigation are prepared by way of Al-Fariss, and Pinder, (1987), Abel-Malek *et al* (2002), Vafai, (2002), Kakac, Kilkis, Kulacki, and Arine, (1991), Rajagopal, Na and Gupta (1984, 1985), and Wafo .(2005). The research is to find a model that is as easy as likely, connecting the minimum number of variables and parameters, and until now including the facility to find out the viscoelastic behavior in compound fluid flows observed by Hulsen (1986, 1990) and Keunings (2003).

A common consent has emerged that the flow with porous media linked with viscoelastic fluids, elastic effects should come up even if their precise nature is unidentified or contentious. Viscoelastic effects in porous media can be imperative insure cases. Whilst in these, the genuine pressure gradient will go beyond the simply viscous gradient further than a serious flow rate, as looked at by some canvassers. It is felled out that the extremely expensive normal stress ratios and differences defined as extensional to shear viscosity related by means of polymeric fluids will produce expanding values of apparent viscosity as flow in the porous media, flow pipes are of quickly changing cross section Viscoelastic fluids have been investigated by Larson (1999) and Sochi (2009, 2010) due to their huge purposes. Oldroyd-B model is the nonlinear viscoelastic model and is a second simplest model and it seems that the most well-liked in fluid flow viscoelastic modeling. Here viscoelastic behaviour will be modelled by the Oldroyd-B (Oldroyd 1958) and Phan-thien / Tanner (PTT) (1977) differential constitutive models and simulation developed by van Os; Phillips. (2004). The result of problems troubled with realistic fluid flow solved with Lie Group techniques has acquired mounting concentration during current years. Lie-Group theory of ODE's and PDE's as a scientific branch created from efforts of the exceptional mathematician Lie of the nineteenth century (1842–1899) and developed by Olver, (1986), Bluman and Kumei (1989), Ibragimov, (1999) and others.and as then it has survived the major constituent part of his most important creation of the continuous groups theory. For PDE's, Lie point symmetries permit the reduction of the number of independent variables expanded by Moran, and Gaggioli, (1968), Abdel-Malek, *et al* (2002), Basov's. (2001), and others.

The correct computations in the fluid dynamics are very imperative, since they influence behavior, safely and economy of complete structure. After finding result,

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it must be checked cautiously. This paper is related with the analytical solutions of viscoelastic Fluid flow in a pipe. Analytical solutions are obtained in the way using generators of the system through Lie Group method and results are checked carefully.

Section 2 is linked with the problem formulation. Section 3 connected with viscoelastic flow solution in circular pipes filled with porous media; section 3.1 associated according to non-homogeneous equation (4-i), section 3.2 concerns with symmetries of the PDE's (13-*a* and c), Section 3.3 combined with invariant solution of PDE's (13-*a* and *c*) corresponding to $X_1 - \alpha X_2$, Section 3.3,1 attached with solution of PDE's (13-b). Finally the conclusions of this paper are identified in section 4.

2. PROBLEM FORMULATIONS

Suppose viscoelastic fluid flow through porous media which is unsteady incompressible laminar flow apprehended in a circular pipe determined in radial direction. A system of cylindrical polar coordinate is related with radius-axis perpendicularly upward. The most important equations system of flow contains of the conservation of both mass and momentum transport related with the Oldroyd–B constitutive model. In the absence of body force adopting Darcy-Brinkman model transfers a system of equations is used. The viscoelastic fluid flow with porous medium is assumed to be homogeneous and isotropic. As the flow in pipe is understood to exist unidirectional expressed within only axial velocity as a function of redial direction along hydro dynamically entirely expanded which velocity does not hinge on the axial route of the pipe and the pressure gradient is believed to exist constant. For unidirectional flow velocity field is given as $\bar{V} = \left(v(r,t),0,0\right)$; wherever the above meaning of velocity mechanically satisfies the incompressibility state. The continuity equation, generalized Darcy–Brinkman model has been employed for the momentum equation through porous media and the Oldroyd–B equation define the stresses of viscoelastic in the fluid flow in vectorial form can be written as under:

$$\nabla.\bar{V} = 0 \qquad \textbf{(1)}$$

$$\frac{\rho}{\varepsilon}\frac{\partial \bar{V}}{\partial t} = \frac{1}{r}\nabla([\frac{\mu_2}{\varepsilon} r \underline{\underline{d}}] + \tau) - \nabla p - \rho \bar{V}.\nabla\bar{V} - \frac{\mu}{K}\bar{V} \qquad \textbf{(2)}$$

The Oldroyd–B constitutive equation describes the viscoelastic stresses in the flow can be expressed as below:

$$\lambda \frac{\partial \tau}{\partial t} = [2\mu_1 \underline{\underline{d}}] - \tau - \lambda\{\bar{V}.\nabla\tau - \nabla\bar{V}.\tau - (\nabla\bar{V})^T.\tau\} \qquad \textbf{(3)}$$

In the above equations, $\bar{V}$ is the velocity vector field of fluid flow, τ is the extra stress tensor, $\underline{\underline{d}}$ is the rate–of–strain tensor, ∇ is the spatial differential operator, *p* is the pressure of isotropic fluid (per unit density) and *t* is the time. The μ_1 is the viscoelastic solute viscosity and μ_2 Newtonian solvent viscosity respectively, fluid density is indicated by ρ, whereas λ is the relaxation time of the viscoelastic fluid and intrinsic permeability of the porous medium is denoted by *K*,. Total viscosity μ of the viscoelastic flow is $\mu = \mu_1 + \mu_2$ and is taken constant and hence porosity of porous media is ε.

The equations are obtained which govern the unsteady unidirectional viscoelastic fluid flow through porous medium accepting Oldroyd–B constitutive model. The velocity field is $\bar{V} = \left(v(r,t),0,0\right)$; here the explanation of velocity automatically gives pleasure to the incompressibility state. The derivation of such equations by employing the momentum transport equation of viscoelastic fluid and Oldroyd–B constitutive equations assuming constant pressure gradient and may be expressed in the absence of body force, the governing system of equations is written in the dimensionless form as under

$$\mathrm{Re}\frac{\partial v}{\partial t} = 1 + \mu_2\frac{\partial^2 v}{\partial r^2} + \frac{\mu_2}{r}\frac{\partial v}{\partial r} + \frac{\partial \tau_{12}}{\partial r} + \frac{\tau_{12}}{r} - \frac{1}{Da}v \qquad \textbf{(i)}$$

$$We\frac{\partial \tau_{11}}{\partial t} = 2We\,\tau_{12}\frac{\partial v}{\partial r} - \tau_{11} \quad \text{(ii) and} \quad We\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial v}{\partial r} - \tau_{12} \quad \textbf{(iii)} \qquad \textbf{(4)}$$

Where *v(r, t)* and $\tau\,(r,t)$ are dimensionless velocity in the axial direction and dimensionless stress tensor in axial, shear and radial direction, *r* is radial coordinates, *t* is the time using for non-dimensional. Where the non-dimensional Reynolds number (R*e*), Weissenberg number (*We*) and Darcy's number (*Da*) are identified as

$$\mathrm{Re} = \frac{R\,\rho\,Vc}{\mu}, \quad We = \frac{\lambda\,Vc}{R}, \; Da = \frac{K}{\varepsilon\,R^2} \text{ and } \mu_1 + \mu_3 = 1$$

As *K* is the adapted permeability concern with the porous medium using for non-dimensional. As *R* is a radius of the pipe and Vc is used for the feature velocity supposed since reference redial velocity $Vc = \left. R^2\left(-\frac{\partial p}{\partial z}\right) \middle/ \mu \right.$

Initial and boundary conditions for completing the well posed problem are taken as

$$v(t,1) = 0, \quad \text{and} \quad \frac{\partial v}{\partial t}(t,0) = 0 \qquad \textbf{When } t > 0 \tag{5}$$

and initial conditions are given as

$$v(0,r) = 0,\ \tau_{11}(0,r) = \tau_{12}(0,r) = 0 \qquad \textbf{When } 0 < r < 1 \tag{6}$$

3. RESULTS OF VISCOELASTIC FLUID FLOW FILLED WITH POROUS MEDIA IN CIRCULAR PIPES

The PDE's system (4) is solved firstly by finding the steady state solution according to non-homogenous equation (4-i) and subject to boundary and initial conditions (5 and 6).

3.1 For the Non homogenous Equation (4-i)

A few problems concerning non–homogeneous equations or boundary conditions can be resolved by means of the transform of dependent variable, $v = v_i + f$

The basic idea to resolve f , a function of one variable, in such a method that v, a function of two variables, is made to satisfy a homogeneous PDEs or homogeneous boundary conditions. Now, we adjust the dependent variables for the non–homogeneous equation (4-i), and to obtain the steady state solution, hence, suppose

$$v(t,r) = v_1(t,r) + f_1(r),\ \tau_{11}(t,r) = v_2(t,r) + f_2(r) \quad \text{and } \tau_{12}(t,r) = v_3(t,r) + f_3(r) \tag{7}$$

Substituting above values in Equation (4), gives the two systems of equations which are that

$$\mu_2\, f_1''(r) + \tfrac{\mu_2}{r}\, f_1'(r) + f_3'(r) + \tfrac{1}{r}\, f_3(r) - \tfrac{1}{Da}\, f_1(r) + 1 = 0 \quad \textbf{(i)} \qquad f_2(r) = 2\,We\ f_3(r)\ f_1'(r) \ \textbf{(ii) and}$$

$$f_3(r) = \mu_1\, f_1'(r) \quad \textbf{(iii)} \tag{8}$$

Subject to boundary conditions $f_1(1) = 0$ **and** $f_1'(0) = 0$ **(9)**

For solving the equations of system (8), set the $f_3(r)$ from (8-iii) in to (8-i), it provides,

$$f_1''(r) + \tfrac{1}{r}\, f_1'(r) - \tfrac{1}{Da}\, f_1(r) + 1 = 0 \tag{10}$$

By using power series solution, solving or integrating the above ODE and applying the boundary conditions, so acquired the result as below

$$f_1(r) = Da\left(1 - \frac{j_0\left(\frac{i\,r}{\sqrt{Da}}\right)}{j_0\left(\frac{i}{\sqrt{Da}}\right)}\right) = Da\left(1 - \frac{\sum\limits_{n=0}^{\infty}(4Da)^{-n}(n!)^{-2}r^{2n}}{\sum\limits_{n=0}^{\infty}(4Da)^{-n}(n!)^{-2}}\right) \tag{11}$$

Here $J_0\left(\frac{i}{\sqrt{Da}}r\right)$ is first kind Bessel function of order zero respectively.

Substitute $f_1(r)$ in equation (8-iii), then $f_3(r)$ is obtained. After replacing with the values of $f_1'(r)$ and $f_3(r)$ in equation (8-ii), then $f_2(r)$ is achieved. Then the following solutions of equations system (8) is completed

$$f_1(r) = Da\left(1 - \frac{J_0(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right), \quad f_2(r) = -2We\,\mu_1\, Da\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right)^2, \quad f_3(r) = i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right) \tag{12}$$

Thus to obtain $v_1(t,r), v_2(t,r)$ and $v_3(t,r)$, the new initial and boundary value problem is given as

$$\mathrm{Re}\ v_{1t} = \mu_2\, v_{1rr} + \frac{\mu_2}{r} v_{1r} + v_{3r} + \frac{v_3}{r} - \frac{1}{Da} v_1 \qquad \textbf{(a)}$$

$$We\ v_{2t} = 2\ We\ \{(v_3 + f_3(r))\, v_{1r} + f_1'(r) v_3\} - v_2 \quad \textbf{(b)} \quad We\ v_{3t} = \mu_1 v_{1r} - v_3 \quad \textbf{(c)} \tag{13}$$

Subject to initial and boundary conditions are,

$$v_1(0,r) = -f_1(r) \quad \textbf{(a)} \quad v_2(0,r) = -f_2(r) \quad \textbf{(b)} \quad v_3(0,r) = -f_3(r) \quad \textbf{(c)} \quad 0 \le r \le 1 \tag{14}$$

$$v_1(t,1) = 0, \quad \textbf{(a)} \quad v_{1r}(0) = 0 \quad \textbf{(b)} \quad t > 0 \tag{15}$$

Where $v_{1t} = \frac{\partial v_1}{\partial t},\ v_{1r} = \frac{\partial v_1}{\partial r},\ v_{1rr},\ v_{3t},\ v_{3r}$, etc, .are partial derivatives.

3.2 Lie Group Analysis of the PDE's (13-*a* and 13- *c*)

Once Lie group algebra of the differential equation is known, it can be employed in the investigation of transformations that will reduce the equation to simpler form and it is powerful method in obtaining analytical solutions of differential equations. In this section, symmetry conditions and method for finding the Lie point symmetries of the above equations (because derivatives of these equations are attached each other) are introduced. The generator

$$X = \phi(t,r,v_1,v_3)\frac{\partial}{\partial t} + \xi(t,r,v_1,v_3)\frac{\partial}{\partial r} + \eta^1(t,r,v_1,v_3)\frac{\partial}{\partial v_1} + \eta^2(t,r,v_1,v_3)\frac{\partial}{\partial v_3} \tag{16}$$

is the Lie point symmetry generator for governed PDE's (13-*a* &13-*c*) iffy,

$$X^{[2]}\left(\mu_2\, v_{1rr} + \frac{\mu_2}{r} v_{1r} + v_{3r} + \frac{v_3}{r} - \frac{1}{Da} v_1 - \mathrm{Re}\ v_{1t}\right)\Big|_{(13-a\,\&\,c)} = 0$$

$$X^{[1]}(We\ v_{3t} - \mu_1 v_{1r} + v_3)\Big|_{(13-a\&c)} = 0$$

Where first and second extended infinitesimal generator of *X* are

$$\textbf{As}\ X^{[1]} = X + \eta_t^{1[1]}\frac{\partial}{\partial u_t} + \eta_r^{1[1]}\frac{\partial}{\partial v_{1r}} + + \eta_t^{2[1]}\frac{\partial}{\partial v_{3t}} + \eta_r^{2[1]}\frac{\partial}{\partial v_{3r}}, \quad X^{[2]} = X^{[1]} + \eta_{rr}^{1[2]}\frac{\partial}{\partial v_{1rr}} + \ldots. \tag{17}$$

In which $\eta_t^{1[1]},\ \eta_r^{1[1]},\ \eta_t^{2[1]},\ \eta_r^{2[1]},\ \eta_{rr}^{1[2]}$ are written by

$$\eta_r^{1[1]} = D_r\eta^1 - v_{1t}\, D_r\phi - v_{1r}\, D_r\xi;\ \eta_t^{1[1]} = D_t\eta^1 - v_{1t}\, D_t\phi - v_{1r}\, D_t\xi\ ;$$

$$\eta_{rr}^{1[2]} = D_r\eta_r^{1[1]} - v_{1tr}\, D_r\phi - v_{1rr}\, D_r\xi\ ;$$

$$\eta_t^{2[1]} = D_t\eta^2 - v_{3t}\, D_t\phi - v_{3r}\, D_t\xi\ ; \qquad \eta_r^{2[1]} = D_r\eta^2 - v_{3t}\, D_r\phi - v_{3r}\, D_r\xi\,;. \tag{18}$$

Where D_t and D_r are the total derivative operators given as

$$D_t = \frac{\partial}{\partial t} + v_{1t}\frac{\partial}{\partial v_1} + v_{1tr}\frac{\partial}{\partial v_{1t}} + v_{1tt}\frac{\partial}{\partial v_{1t}} + v_{3t}\frac{\partial}{\partial v_3} + v_{3r}\frac{\partial}{\partial v_{3tr}} + v_{3tt}\frac{\partial}{\partial v_{3t}} + v_{3tr}\frac{\partial}{\partial v_{3r}} + \ldots\ldots\ldots\ldots,$$

$$D_r = \frac{\partial}{\partial r} + v_{1r}\frac{\partial}{\partial v_1} + v_{1tr}\frac{\partial}{\partial v_{1t}} + v_{1rr}\frac{\partial}{\partial v_{1r}} + v_{3r}\frac{\partial}{\partial v_3} + v_{3t}\frac{\partial}{\partial v_{3tr}} + v_{3rr}\frac{\partial}{\partial v_{3r}} + v_{3tr}\frac{\partial}{\partial v_{3t}} + \ldots\ldots\ldots\ldots, \tag{19}$$

As $X^{[2]}\left(\mu_2 v_{1rr} + \frac{\mu_2}{r} v_{1r} + v_{3r} + \frac{v_3}{r} - \frac{1}{Da} v_1 - \mathrm{Re}\, v_{1t}\right)\Big|_{(13-a\,\&\,c)} = 0$

$$\Rightarrow -\frac{1}{r^2}(\mu_2 v_{1r} + v_3)\xi - \frac{1}{Da}\eta^1 + \frac{1}{r}\eta^2 - \mathrm{Re}\,\eta_t^{1[1]} + \frac{\mu_2}{r}\eta_r^{1[1]} + \eta_r^{2[1]} + \mu_2\eta_{rr}^{1[2]}\Big|_{(13-a\,\&\,c)} = 0 \tag{20}$$

$$X^{[1]}(We\, v_{3t} - \mu_1 v_{1r} + v_3)\Big|_{(13-a\&c)} = 0 \Rightarrow \eta^2 - \mu_1\eta_r^{1[1]} + We\,\eta_t^{2[1]}\Big|_{(13-a\&c)} = 0 \tag{21}$$

Where $X^{[1]}$, $X^{[2]}$ and ($\eta_t^{1[1]}, \eta_r^{1[1]}, \eta_{rr}^{1[2]}, \eta_t^{2[1]}, \eta_r^{2[1]}$) are stated in the relations (18 and 19). In the generator X and the above equations (20 and 21), according to Lie group theory, the unknown functions $\phi, \xi, \eta^1, \eta^2$ are taken independent of the differentials of the primitive variables v_1 and v_3 and established from the determining equations derived from the invariance condition. Thus separating and equating them by the derivatives of v_1 and v_3 and their powers deals to the two simplified over resolved systems of PDE's and after solving these two over determined systems of linear PDE's gives rise to the values of the unidentified functions ϕ, ξ, η^1 and η^2 as

$$\phi = c_1,\ \xi = 0, \qquad \eta^1 = c_2 v_1 + g(t,r) \ \textbf{and}\ \eta^3 = c_2 v_3 + h(t,r) \tag{22}$$

Here $g(t,r)$ and $h(t,r)$ are arbitrary functions of r of the following partial differential equations.

$$\mathrm{Re}\frac{\partial g}{\partial t} = \mu_2\frac{\partial^2 g}{\partial r^2} + \frac{\mu_2}{r}\frac{\partial g}{\partial r} + \frac{\partial h}{\partial r} + \frac{h}{r} - \frac{1}{Da}g \qquad \textbf{and} \qquad We\frac{\partial h}{\partial t} = \mu_1\frac{\partial g}{\partial r} - h$$

In (22), c_1 and c_2 are constants of integration. Thus the symmetry Lie algebra of the PDEs (13-*a*) and (13-*c*) is two–dimensional and identified by the following generators:

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = v_1\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_3} \quad \textbf{and}\ X_m = g(t,r)\frac{\partial}{\partial v_1} + h(t,r)\frac{\partial}{\partial v_3} \tag{23}$$

Where m is any natural number

3.3 Invariant Solution of the PDEs (13-*a* and *c*) corresponding to Operator $X_1 - \beta X_2$

The form of invariant result related in the generator $X = X_1 - \beta X_2$ is given as

$$v_1(t,r) = e^{-\beta t}\varphi(r) \quad \textbf{and}\quad v_3(t,r) = e^{-\beta t}\phi(r) \tag{24}$$

For bounded function, we must take exponential function in negative sign.
After putting the values of (25) into PDEs (13-*a* and13-*c*) which gives the reduced ODEs system

$$\mu_2\varphi''(r) + \frac{\mu_2}{r}\varphi'(r) + \phi'(r) + \frac{1}{r}\phi(r) + \left(\beta\,\mathrm{Re} - \frac{1}{Da}\right)\varphi(r) = 0 \quad (\boldsymbol{a}) \qquad \phi(r) = \frac{\mu_1}{1-\beta We}\varphi'(r) \quad (\boldsymbol{b}) \tag{25}$$

Put the value of $\phi(r)$ from (25-b) into (25-*a*), then we have,

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + \lambda^2\varphi(r) = 0 \qquad \textbf{Where}\ \lambda^2 = \frac{(\beta\,\mathrm{Re} - \frac{1}{Da})(1-\beta We)}{(1-\beta\mu_2 We)} \tag{26}$$

Subject to boundary conditions $\varphi(1)=0$ **and** $\varphi'(0)=0$ **(27)**

The above PDE (26) is the Bessel's differential equation of order zero and similarly equation have been solved as in section 3.1 by using power series solution and then the general solution of Bessel's differential equation is

$$\varphi(r)=c_1\,J_0(\lambda r)+\;c_2\,Y_0(\lambda r) \tag{28}$$

Where $J_0(\lambda r)$ and $Y_0(\lambda r)$ are Bessel function of order zero of first and second kind respectively, i-e.

$$J_0(\lambda r)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2\,2^{2n}}\,(\lambda\;r)^{2n}\;\textbf{and}\qquad Y_0\,(r)=\frac{2}{\pi}[\,y_2(r)\,+\,(\gamma\,+\,\ln(2)\,j_0(\lambda\,r)$$

$$y_2\,(r)=\ln(r)\,j_0(\lambda r)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\psi\,(n)}{2^{2n}\,(n!)^2}(\lambda r)^{2n} \tag{29}$$

Where γ is the co-efficient of combination and $\psi(n)=[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+........+\frac{1}{n}]$ for n=1, 2, 3, 4......

As $Y_0(\lambda r)\to-\infty$ when $r=0$, so it is neglected, Therefore, the result is in one bounded solution is given as $\varphi(r)=c_1\,J_0(\lambda r)$

From equation (27), after applying the boundary conditions so $\varphi'(0)=0$ is identically satisfied and

$\varphi(1)=c_1\,J_0(\lambda)=0\Rightarrow c_1\neq 0$, **so** $J_0(\lambda)=0$,

It has infinite number of roots $\lambda_n\;(n=1,2,3,.....,\infty)$. Therefore after applying the superposition principle, we acquire the result

$$\varphi(r)=\sum_{n=1}^{\infty}c_n\;J_0(\lambda_n\,r) \tag{30}$$

and replacement the value of $\varphi(r)$ in equation (25-*b*), then $\phi(r)$ is achieved i-e

$$\phi(r)=\sum_{n=1}^{\infty}\frac{-\mu_1\,\lambda_n\,c_n}{(1-\beta We)}\,J_1(\lambda_n\,r) \tag{31}$$

As $\lambda^2\;=\dfrac{(\beta\,\mathrm{Re}-\frac{1}{Da})(1-\beta We)}{(1-\beta\,\mu_2\,We)}\Rightarrow\beta=\dfrac{1}{2}(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\;\;\mathrm{Re}})\pm\frac{1}{2}\sqrt{(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\;\;\mathrm{Re}})^2-\frac{4(\,Da\;\lambda_n^2+1)}{4(\,Da\;\lambda_n^2+1)}}$

As answers are got after joint two equations and have same boundary points, so for the time function, assume

$$\beta_1\;=\frac{1}{2}(\frac{1}{We}+\frac{\mu_2\;\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})+\frac{1}{2}\sqrt{(\frac{1}{We}+\frac{\mu_2\;\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})^2-\frac{4\,(\,Da\,\lambda_n^2+1)}{Da\,\mathrm{Re}\,We}} \tag{32-a}$$

$$\beta_2\;=\frac{1}{2}(\frac{1}{We}+\frac{\mu_2\;\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})-\frac{1}{2}\sqrt{(\frac{1}{We}+\frac{\mu_2\;\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})^2-\frac{4\,(\,Da\,\lambda_n^2+1)}{Da\,\mathrm{Re}\,We}} \tag{32-b}$$

Therefore, equation (24) develops into as below:

$$v_1(t,r)=\sum_{n=1}^{\infty}\Big(c_{n1}\,e^{-\beta_1 t}+c_{n2}e^{-\beta_2 t}\Big)J_0(\lambda_n\,r)\;\;\textbf{and}\;\;v_3(t,r)=\sum_{n=1}^{\infty}-\mu_1\,\lambda_n\left(\frac{c_{n1}\,e^{-\beta_1 t}}{(1-\beta_1\,We)}+\frac{c_{n2}e^{-\beta_2 t}}{(1-\beta_2\,We)}\right)J_1(\lambda_n\,r) \tag{33}$$

For the constants, applying the initial conditions (14-*a*) and (14-*c*), so we obtain

$$c_{n\,1}=\frac{2\,Da\;\;(1-\beta_1\,We)\,\beta_2}{(\beta_1-\beta_2)\,\lambda_n\,(\,Da\;\lambda_n^{\,2}+1)\,J_1(\lambda_n)}\qquad\textbf{and}\;c_{n\,2}=\frac{-2\,Da\;\;(1-\beta_2\,We)\,\beta_1}{(\beta_1-\beta_2)\,\lambda_n\,(\,Da\;\lambda_n^{\,2}+1)\,J_1(\lambda_n)}$$

Set the values c_{n1} and c_{n2} in the relation (33), we obtain

$$v_1(t,r)=\sum_{n=1}^{\infty}\frac{2\,Da}{\lambda_n\,(Da\,\lambda_n^2-1)\,J_1(\lambda_n)}\left(\frac{(1-\beta_1 We)\,\beta_2\, e^{-\beta_1 t}}{(\beta_1-\beta_2)}-\frac{(1-\beta_2 We)\,\beta_1\, e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right)J_0(\lambda_n\, r) \qquad \textbf{(34-}\textbf{\textit{a}}\textbf{)}$$

$$v_3(t,r)=\sum_{n=1}^{\infty}\frac{-2\,\mu_1 Da}{(Da\,\lambda_n^2-1)\,J_1(\lambda_n)}\left(\frac{\beta_2\, e^{-\beta_1 t}}{(\beta_1-\beta_2)}-\frac{\beta_1\, e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right)J_1(\lambda_n\, r) \qquad \textbf{(34-}\textbf{\textit{b}}\textbf{)}$$

3.3.1 Solution of Partial Differential Equation (13-*b*)

Hence PDE is (13-*b*) is given as $We\, v_{2t}=2\, We\,\{(v_3+f_3(r))\,v_{1r}+f_1'(r)\,v_3\}-v_2$

As $f_3(r)=i\,\mu_1\sqrt{Da}\left(\dfrac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right)$ **,Consider** $J_1(\frac{i}{\sqrt{Da}}r)=\sum_{n=1}^{\infty}c_n\, J_1(\lambda_n r)\Rightarrow c_n=\dfrac{2i\sqrt{Da}\,J_0(\frac{i}{\sqrt{Da}})}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}$ **When** $J_0(\lambda_n)=0$

So we have $J_1(\dfrac{i\,r}{\sqrt{Da}})=\sum_{n=1}^{\infty}\dfrac{2i\sqrt{Da}\,J_0(\frac{i}{\sqrt{Da}})}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\,J_1(\lambda_n\, r)$

After substituting the value of $f_3(r)$, $f_1'(r)$ and according to the equation (34-*a* and *b*), solution of PDE (13-*b*) is arranged as

$$v_2(t,r)=\left(\sum_{n=1}^{\infty}\frac{Da\,(8\,We\,\mu_1)^{\frac{1}{2}}}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\begin{pmatrix}\dfrac{\beta_2^2(1-\beta_1 We)\,e^{-2\beta_1 t}}{(\beta_1-\beta_2)^2\,(1-2\beta_1 We)}+\dfrac{\beta_1^2(1-\beta_2 We)\,e^{-2\beta_2 t}}{(\beta_1-\beta_2)^2\,(1-2\beta_2 We)}+\dfrac{\beta_2\,(2-\beta_1 We)\,e^{-\beta_1 t}}{((\beta_1-\beta_2)(1-\beta_1 We)}\\ -\dfrac{\beta_1\,(2-\beta_2 We)\,e^{-\beta_2 t}}{(\beta_1-\beta_2)(1-\beta_2 We)}-\dfrac{\beta_1\beta_2(2-\beta_1 We-\beta_2 We)\,e^{-(\beta_1-\beta_2)t}}{(\beta_1-\beta_2)^2(1-\beta_1 We-\beta_2 We)}\end{pmatrix}^{\frac{1}{2}} J_1(\lambda_n r)\right)^2+e^{\frac{-t}{We}}\,\omega(r)$$

Now applying the initial condition (14-*b*), then we obtain,

$$\omega(r)=2We\,\mu_1\,Da\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)^2-\left(\sum_{n=1}^{\infty}\frac{Da\,(8\,We\,\mu_1)^{\frac{1}{2}}}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\begin{pmatrix}\dfrac{\beta_2^2(1-\beta_1 We)\,e^{-2\beta_1 t}}{(\beta_1-\beta_2)^2\,(1-2\beta_1 We)}+\dfrac{\beta_1^2(1-\beta_2 We)\,e^{-2\beta_2 t}}{(\beta_1-\beta_2)^2\,(1-2\beta_2 We)}+\dfrac{\beta_2\,(2-\beta_1 We)\,e^{-\beta_1 t}}{((\beta_1-\beta_2)(1-\beta_1 We)}\\ -\dfrac{\beta_1\,(2-\beta_2 We)\,e^{-\beta_2 t}}{(\beta_1-\beta_2)(1-\beta_2 We)}-\dfrac{\beta_1\beta_2(2-\beta_1 We-\beta_2 We)\,e^{-(\beta_1-\beta_2)t}}{(\beta_1-\beta_2)^2(1-\beta_1 We-\beta_2 We)}\end{pmatrix}^{\frac{1}{2}} J_1(\lambda_n r)\right)^2 \qquad \textbf{(35)}$$

Therefore, final result of the PDE's system (4 to 6) accept the following solutions

$$v(t,r)=\sum_{n=1}^{\infty}\frac{2\,Da}{\lambda_n\,(Da\,\lambda_n^2-1)\,J_1(\lambda_n)}\left(\frac{(1-\beta_1 We)\,\beta_2\, e^{-\beta_1 t}}{(\beta_1-\beta_2)}-\frac{(1-\beta_2 We)\,\beta_1\, e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right)J_0(\lambda_n r)+Da\left(1-\frac{J_0(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) \qquad \textbf{(36-}\textbf{\textit{a}}\textbf{)}$$

$$\tau_{11}(t,r)=\left(\sum_{n=1}^{\infty}\frac{Da\,(8We\,\mu_1)^{\frac{1}{2}}}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\begin{pmatrix}\dfrac{\beta_2^{\,2}(1-\beta_1 We)\,(e^{-2\beta_1 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2\,(1-2\beta_1 We)}+\dfrac{\beta_1^{\,2}(1-\beta_2 We)\,(e^{-2\beta_2 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2\,(1-2\beta_2 We)}\\ -\dfrac{\beta_1\,\beta_2(2-\beta_1 We-\beta_2 We)\,(e^{-(\beta_1+\beta_2)t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2\,(1-\beta_1 We-\beta_2 We)}\\ +\dfrac{\beta_2\,(2-\beta_1 We)\,(e^{-\beta_1 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)\,(1-\beta_1 We)}-\dfrac{\beta_1\,(2-\beta_2 We)\,(e^{-\beta_2 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)\,(1-\beta_2 We)}\end{pmatrix}^{\frac{1}{2}} J_1(\lambda_n r)\right)^2 \qquad \textbf{(36-}\textbf{\textit{b}}\textbf{)}$$

$$+\;2\,We\,\mu_1\,Da\left(e^{\frac{-t}{We}}-1\right)\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)$$

$$\tau_{12}(t,r)=\sum_{n=1}^{\infty}\frac{-2\,\mu_1 Da}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\left(\frac{\beta_2\, e^{-\beta_1 t}}{(\beta_1-\beta_2)}-\frac{\beta_1\, e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right)J_1(\lambda_n r)+i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) \qquad \textbf{(36-}\textbf{\textit{c}}\textbf{)}$$

4. CONCLUSIONS

The point of this paper was to develop mathematical models and to find the invariant solutions of the PDE's system arising in the study of viscoelastic fluid flow in pipes filled with porous media using with Oldroyd–B Constitutive Model. Lie group method is applied successfully to obtain the invariant solutions of the problem. The Lie group is a theoretic approach which is applied to determine the solutions of the problem. The one number of independent variables has been reduced through one-parameter group of transformation and the PDE's system reduces to an ODE's system and the invariant solutions are acquired. The purpose of the investigation is to find the exact analytical result of velocity accepting Lie group technique. We wish that the results may be helpful for other researchers in this field. Our recommendations for the future work are expanding and leaving into practice other steady-state and transient viscoelastic algorithms.

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