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An Analysis of Stresses and Applications of Viscoelastic Fluid Flow in a circular Pipe through Porous media via with Oldroyd–B Constitutive Model

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Abstract: In the field of computational fluid dynamics, finding the analytical solutions of partial differential equations governing complex problems presents a challenge to the researchers. Our basic aim is to find the symmetries of Lei- group theory of viscoelastic fluid flows to obtain the results to the PDE's system including the continuity equation, momentum equations and constitutive equations related with viscoelastic stresses, under suitable initial and boundary conditions.This paper presents the passing hydrodynamics actions of the viscoelastic fluid flow in pipes through porous media related with Darcy-Brinkman model. The research is lectured by means of analytical results and also numerical results of the problems of given differential equations associated by means of initial and boundary conditions, happening in the knowledge of viscoelastic flow in pipes through porous media. The solutions of problem of this paper are obtained adopting the Lie group of transformation theoretic approach and analytical solution is obtained under the symmetries for the equations through Lie group technique.

Keywords: Viscoelastic Flow in a pipe, porous Media, Darcy-Brinkman model, Oldroyd–B constitutive model, Lei- group theory, Analysis of Viscoelastic Stresses

1. INTRODUCTION

Newtonian and non-Newtonian fluids are expressively concerned large to interest in the literature. Material of viscoelastic can be prospected as the transitional states among the viscous fluids and elastic solids. The substance displays behavior of elastic, such as the effects of memory, as well as the properties of fluid. Viscoelasticity can be modeled by connecting Newton's rule for fluids of viscous (here stress is directly proportional to strain rate) by means of Hook's law used for elastic solids, since presented with the original Maxwell model and enlarged with the Convicted Maxwell models used for the viscoelastic nonlinear fluids. Much difficult rheological and hydrodynamic behavior of compound fluids can be considered like effects of the inner elastic possessions. The governing equations of the problem are related with the viscoelastic flow that illustrates partial elastic recovery upon the removal of a deforming stress. As the constitutive model itself is simple, the dynamics that occur in simple flows are difficult and present a considerable challenge to numerical simulation, due to infinite extensional viscosity at finite elongation–rate. Viscoelastic fluids have been researched due to their vast applications for some decades to know the phenomena related with it and investigated by Taha Sochi (2009, 2010), Larson (1999), Rajagopal and. Gupta, (1984), Rajagopal and Na, (1985) and Keunings, (2003),. The investigation is to derive a model that is as easy as likely, relating the minimum number of parameters and variables, and until now containing the facility to determine the viscoelastic activities in compound fluid flows.

The objective of this paper is to present investigative solutions for viscoelastic fluid flows in pipes through porous medium adopting the viscosity which is constant related with Oldroyd–B constitutive model. Oldroyd-B model is the nonlinear viscoelastic model and is a second simplest model and it seems that the most well-liked in fluid flow viscoelastic modeling. Here viscoelastic behaviour will be modelled by the Oldroyd-B (Oldroyd 1958) and Phan-thien/Tanner (PTT) (1977) differential constitutive models and simulation developed by van Os; Phillips. (2004) and Wafo (2005).

This investigation is concerned with how to find and use symmetries for partial differential equations (PDE's); symmetry of PDE's maps any solution of PDE's to another solution of the same PDE's. The problem is to find and use admitted Lie point symmetries algebra. Symmetries of differential equation are forming a confined one-parameter group of transformation and depending continuously on a parameter can be designed algorithmically by Sophus Lie (1842–1899) and developed by Bluman and Kumei (1989), Olver, (1986), Ibragimov, (1999) and others.

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These algebras are made use of to reduce the governing PDE's system to solvable form. Basically, to determine an admitted Lie point symmetry one must consider transformations, acting on spaces of a finite number of variables, which depart invariant the solution multiple of the given PDE's and its differential consequences. For PDE's, symmetries allow the reduction of the number of independent variables developed by Moran, and Gaggioli, (1968), Abdel-Malek, Badran and Hassan (2002), Basov's. (2004) and others. The solution of the governing PDE's system is acquired analytically or numerically in the way using symmetries of the system through symmetry method. Numerical predictions of a system are resolute adopting Mathematica solver ND-Solve.

Section 2 concerns with the formulation in mathematics. Section 3 associated with viscoelastic flow solution in circular pipes through porous media; section 3.1 connected with study state solution for non-homogeneous equation, section 3.2 consists of solution of PDE's system (10) to (12) using symmetry method. Lie-point symmetries of the PDE's (10-i & 10-iii) of viscoelastic flow in pipes are determined in section 3.2. Section 3.2.2 related with invariant solution corresponding to $X_1 - \alpha X_2$, result of PDE's (10-ii) linked with the section 3.2.3. Section 4 is related analysis of Viscoelastic stresses. As section 4.1 concerned with analytical solutions of normal and shear stresses. Steady state solution of normal and shear stresses and its graphs is discussed in section 4.2. As Section 4.3 connected with numerical solution of stresses of viscoelastic fluid flow in pipes filled with porous space. As section 5 associated with conclusions of the problems.

2. PROBLEM SPECIFICATION

Consider a tubular porous layer held in a pipe drenched with the incompressible laminar flow of viscoelastic fluid in radial direction. A polar coordinate system is applied with radius-axis vertically upward. The main equations system of flow includes of the conservation of momentum and conservation of mass transport paired with the Oldroyd–B constitutive model.

The viscoelastic fluid flow in the course of porous medium is believed to possess isotropic and homogeneous. For unidirectional flow velocity field is given as $\bar{u} = \left(u(r,t), 0, 0\right)$; wherever the above meaning of velocity mechanically satisfies the incompressibility state. The generalized Darcy–Brinkman model has been employed for the momentum equation and if body force is not present; continuity equation, generalised equation of momentum through porous media and the Oldroyd–B equation defines the stresses of viscoelastic in the fluid flow in vectorial form can be given as:

$$\nabla . \bar{u} = 0 \tag{1}$$

$$\frac{\rho}{\varepsilon}\frac{\partial \bar{u}}{\partial t} = \frac{1}{r}\nabla([\frac{\mu_2}{\varepsilon} r \underline{\underline{d}}] + \tau) - \nabla p - \rho \bar{u} . \nabla \bar{u} - \frac{\mu}{K}\bar{u} \text{ ,} \tag{2}$$

where as t represents time so that $\frac{\partial}{\partial t}$ is a temporal derivative, ρ and μ is the fluid density and total viscosity of viscoelastic fluid respectively, $.\bar{u}$ is used for the field of velocity vector, τ is the extra stress tensor, $\underline{\underline{d}}$ is the rate–of–strain tensor, ∇ represents a spatial operator for differential, μ_2 is denoted for the Newtonian solvent viscosity, p is the isotropic fluid pressure, the intrinsic permeability within the porous media is identified with K and ε is porosity of porous media.

The constitutive equation of Oldroyd–B model defines the stresses of viscoelastic in the fluid flow may be written in the form as under,

$$\lambda \frac{\partial \tau}{\partial t} = [2\mu_1 \underline{\underline{d}}] - \tau - \lambda\{ \bar{u} . \nabla \tau - \nabla \bar{u} . \tau - (\nabla \bar{u})^T . \tau \} \tag{3}$$

Where the rest time for the fluid of viscoelastic is indicated by λ and μ_1 is used for viscoelastic solute viscosity. As total viscosity is $\mu = \mu_1 + \mu_2$ and is taken constant.

The equations are derived which govern the unsteady unidirectional flow of viscoelastic fluid through porous media adopting of constitutive equation within an Oldroyd-B flow fluid, the equations are obtained by leading the unidirectional fluid flow of viscoelastic in porous pipes. The derivation of these equations by employing the transport equation of momentum and constitutive equations of Oldroyd–B suppose that pressure gradient is constant and body force is not present, the following dimensionless form equations in the present problem are considered for mathematical modeling.

$$\mathrm{Re}\frac{\partial u}{\partial t}=1+\mu_2\frac{\partial^2 u}{\partial r^2}+\frac{\mu_2}{r}\frac{\partial u}{\partial r}+\frac{\partial\tau_{12}}{\partial r}+\frac{\tau_{12}}{r}-\frac{1}{Da}u \qquad \text{(i)}$$

$$We\frac{\partial\tau_{11}}{\partial t}=2We\,\tau_{12}\frac{\partial u}{\partial r}-\tau_{11} \quad \text{(ii) and} \quad We\frac{\partial\tau_{12}}{\partial t}=\mu_1\frac{\partial u}{\partial r}-\tau_{12} \quad \text{(iii)} \qquad (4)$$

Where u(r, t) and $\tau(r,t)$ are dimensionless velocity in the axial direction and dimensionless stress tensor in axial, shear and radial direction, *r* is radial coordinates, *t* is the time using for non-dimensional and the dimensionless Reynolds number (*Re*), Weissenberg number (*We*) and Darcy's number (*Da*) are defined as

$$\mathrm{Re}=\frac{R\rho Vc}{\mu},\quad We=\frac{\lambda Vc}{R},\quad Da=\frac{K}{\varepsilon R^2}\ \text{and}\ \mu_1+\mu_3=1$$

Hence K is the adapted permeability concern with the porous medium using for non-dimensional. As *R* is a radius of the pipe and Vc is used for the feature velocity supposed since reference redial velocity $Vc=\dfrac{\varepsilon R^2\left(-\frac{\partial p}{\partial z}\right)}{\mu}$

It is necessary to prescribe initial and boundary conditions for to complete the well posed problem specification,. So initial conditions are given as: $u(t,1)=0,$ and $\dfrac{\partial u}{\partial t}(t,0)=0$ When t > 0 (5)

and initial conditions are taken as $u(0,r)=\tau_{11}(0,r)=\tau_{12}(0,r)=0$ When $0<r<1$ (6)

3. Viscoelastic flow Solutions in circular pipes through porous media

The PDE's system (4) subject to initial and boundary conditions (5 & 6) is solved by finding the firstly steady state solution for non-homogenous equation (4-i)

3.1 Steady State Solution for Non-homogenous equation (4-i)

Non–homogeneous equations can be solved by means of a change of dependent variable and to find the steady state solution, hence, consider

$$u(t,r)=v_1(t,r)+\varphi_1(r),\ \tau_{11}(t,r)=v_2(t,r)+\varphi_2(r)\,\&\,\tau_{12}(t,r)=v_3(t,r)+\varphi_3(r) \quad (7)$$

Substituting these values in Equation (4), and separating the like terms of one and two dependent variables, gives the two systems of equations which are

$$\mu_2\varphi_1''(r)+\frac{\mu_2}{r}\varphi_1'(r)+\varphi_3'(r)+\frac{\varphi_3(r)}{r}-\frac{1}{Da}\varphi_1(r)+1=0 \quad \text{(i)} \qquad \varphi_2(r)=2We\,\varphi_3(r)\varphi_1'(r) \ \text{(ii) \&}$$

$$\varphi_3(r)=\mu_1\varphi_1'(r) \qquad \text{(iii)} \qquad (8)$$

Subject to boundary conditions $\varphi_1(1)=0$ and $\varphi_1'(0)=0$ (9)

Thus to determine $v_1(t,r), v_2(t,r), v_3(t,r)$, the new boundary value problem is given as

$$\mathrm{Re}\frac{\partial v_1}{\partial t}=\mu_2\frac{\partial^2 v_1}{\partial r^2}+\frac{\mu_2}{r}\frac{\partial v_1}{\partial r}+\frac{\partial v_3}{\partial r}+\frac{v_3}{r}-\frac{1}{Da}v_1 \qquad \text{(i)}$$

$$We\frac{\partial v_2}{\partial t}=2We\{(v_3+\varphi_3(r))\frac{\partial v_1}{\partial r}+\varphi_1'(r)v_3\}-v_2 \quad \text{(ii)} \quad We\frac{\partial v_3}{\partial t}=\mu_1\frac{\partial v_1}{\partial r}-v_3 \quad \text{(iii)} \qquad (10)$$

Subject to initial and boundary conditions are,

$$v_1(t,1)=0, \quad \text{(i)} \qquad \frac{\partial v_1(t,0)}{\partial r}=0 \quad \text{(ii)} \qquad t>0\ \& \qquad (11)$$

$$v_1(0,r)=-\varphi_1(r) \ \text{(i)} \quad v_2(0,r)=-\varphi_2(r) \ \text{(ii)} \quad v_3(0,r)=-\varphi_3(r) \ \text{(iii)} \quad 0\le r\le 1 \qquad (12)$$

For solving the system of equations (8), putting the $\varphi_3(r)$ from (8-iii) in to (8-i), then differential equation is obtained as

$$\varphi_1''(r)+\frac{1}{r}\varphi_1'(r)+\frac{i^2}{Da}\varphi_1(r)+1=0 \qquad (13)$$

So for C.F. of equation (13), we have

$$\varphi_1''(r)+\frac{1}{r}\varphi_1'(r)+\frac{i^2}{Da}\varphi_1(r)=0$$

The above equation is Bessel's differential equation of order zero. As general result of the (14) is written as

$$\varphi_{1c}(r)=A\,J_0(\tfrac{i}{\sqrt{Da}}r)+BY_0(\tfrac{i}{\sqrt{Da}}r)$$

Where $J_0(\frac{i}{\sqrt{Da}}r)$ and $Y_0(\frac{i}{\sqrt{Da}}r)$ are Bessel function of order zero of first and second kind, respectively. Of course, equation (14) is singular or $Y_0(\frac{i}{\sqrt{Da}}r)\to-\infty$ when $r=0$. Physically meaningful solution must be twice continuously differentiable in $0\le r\le 1.$ so, we must take $B=0$ and equation (14) has only one bounded solution, i-e $\varphi_{1c}(r)=A\,J_0(\frac{i}{\sqrt{Da}}r)$

As particular solution of equation (13) is given as $\varphi_{1p}(r)=Da$

Hence solution of equation (13) is obtained as

$$\varphi_1(r)=\varphi_{1c}(r)+\varphi_{1p}(r)=A\,J_0(\tfrac{i}{\sqrt{Da}}r)+Da \tag{14}$$

Now, applying the boundary conditions (9), then we obtain $A=\frac{-Da}{J_0(\frac{i}{\sqrt{Da}})}$. The condition $\varphi_1'(0)=0$ is identically satisfied. So

$$\varphi_1(r)=Da\left(1-\frac{J_0(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right) \tag{15}$$

Substitute this value of $\varphi_1(r)$ in equation (8-iii), then $\varphi_3(r)$ is obtained. After substituting the values of $\varphi_1'(r)$ and $\varphi_3(r)$ in equation (8-ii), then $\varphi_2(r)$ is obtained. Then the following solution of system of equations (8) is achieved

$$\varphi_1(r)=Da\left(1-\frac{J_0(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right),\ \varphi_2(r)=-2We\,\mu_1\,Da\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right)^2,\ \varphi_3(r)=i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right) \tag{16}$$

3.2 Solution of system of PDE's (10) to (12) using symmetry method

3.2.1 Lie-point symmetries of the PDE's (10-i) & (10- iii) of viscoelastic flow in pipes

In this section, symmetry conditions and method for finding the Lie point symmetries of the equations (10-i &10-iii) are introduced because derivatives of these PDEs are linked each other and Let us the one parameter Lie point transformations of (t,r,v_1,v_3) is considered by

$$t^*=t+\delta\phi(t,r,v_1,v_3)+.........., \qquad r^*=r+\delta\xi(t,r,v_1,v_3)+..........,$$

$$v_1^*=v_1+\delta\eta^1(t,r,v_1,v_3)+........., \qquad v_3^*=v_3+\delta\eta^2(t,r,v_1,v_3)+........, \tag{17}$$

The system of PDE's (10-i & iii) is invariant under the transformation (21) if it is invariant under the generator

$$X=\phi(t,r,v_1,v_3)\frac{\partial}{\partial t}+\xi(t,r,v_1,v_3)\frac{\partial}{\partial r}+\eta^1(t,r,v_1,v_3)\frac{\partial}{\partial v_1}+\eta^2(t,r,v_1,v_3)\frac{\partial}{\partial v_3} \tag{18}$$

The unknown functions $\phi,\ \xi,\ \eta^1,\ \eta^2$ are established from the determining equations derived from the invariance condition.

As $X^{[1]}=X+\eta_t^{1[1]}\frac{\partial}{\partial u_t}+\eta_r^{1[1]}\frac{\partial}{\partial v_{1r}}++\eta_t^{2[1]}\frac{\partial}{\partial v_{3t}}+\eta_r^{2[1]}\frac{\partial}{\partial v_{3r}}, \qquad X^{[2]}=X^{[1]}+\eta_{rr}^{1[2]}\frac{\partial}{\partial v_{1rr}}+....$ (19)

be the corresponding first and second extended infinitesimal generator for the governed PDEs ((10-i &10-iii), Where $\eta_t^{1[1]},\ \eta_r^{1[1]},\ \eta_t^{2[1]},\ \eta_r^{2[1]},\ \eta_{rr}^{1[2]}$ are written by

$$\eta_t^{1[1]}=D_t\eta^1-v_{1t}\,D_t\phi-v_{1r}\,D_t\xi\ ; \qquad \eta_r^{1[1]}=D_r\eta^1-v_{1t}\,D_r\phi-v_{1r}\,D_r\xi;$$

$$\eta_t^{2[1]}=D_t\eta^2-v_{3t}\,D_t\phi-v_{3r}\,D_t\xi\ ;$$

$$\eta_r^{2[1]} = D_r\eta^2 - v_{3t}\, D_r\phi - v_{3r}\, D_r\xi\,; \qquad \eta_{rr}^{1[2]} = D_r\eta_r^{1[1]} - v_{1tr}\, D_r\phi - v_{1rr}\, D_r\xi\,. \tag{20}$$

Where D_t and D_r are the total derivative operators given as

$$D_t = \frac{\partial}{\partial t} + v_{1t}\frac{\partial}{\partial v_1} + v_{1tr}\frac{\partial}{\partial v_{1t}} + v_{1tt}\frac{\partial}{\partial v_{1t}} + v_{3t}\frac{\partial}{\partial v_3} + v_{3r}\frac{\partial}{\partial v_{3tr}} + v_{3tt}\frac{\partial}{\partial v_{3t}} + v_{3tr}\frac{\partial}{\partial v_{3r}} + \ldots\ldots\ldots.,$$

$$D_r = \frac{\partial}{\partial r} + v_{1r}\frac{\partial}{\partial v_1} + v_{1tr}\frac{\partial}{\partial v_{1t}} + v_{1rr}\frac{\partial}{\partial v_{1r}} + v_{3r}\frac{\partial}{\partial v_3} + v_{3t}\frac{\partial}{\partial v_{3tr}} + v_{3rr}\frac{\partial}{\partial v_{3r}} + v_{3tr}\frac{\partial}{\partial v_{3t}} + \ldots\ldots\ldots., \tag{21}$$

Therefore one parameter Lie group of transformations (21 to 24) is admitted by the governed PDEs (10-i &10-iii).iffy

$$X^{[2]}\left(\mu_2 v_{1rr} + \frac{\mu_2}{r}v_{1r} + v_{3r} + \frac{v_3}{r} - \tfrac{1}{Da}v_1 - \text{Re}\, v_{1t}\right)\Bigg|_{(10-i\,\&\,iii)} = 0$$

$$\Rightarrow\ -\frac{1}{r^2}(\mu_2 v_{1r} + v_3)\xi - \tfrac{1}{Da}\eta^1 + \frac{1}{r}\eta^2 - \text{Re}\,\eta_t^{1[1]} + \frac{\mu_2}{r}\eta_r^{1[1]} + \eta_r^{2[1]} + \mu_2\eta_{rr}^{1[2]}\Bigg|_{(10-i\,\&\,iii)} = 0 \tag{22-i}$$

$$X^{[1]}(We\, v_{3t} - \mu_1 v_{1r} + v_3)\Big|_{(10-i\&iii)} = 0 \Rightarrow \eta^2 - \mu_1\eta_r^{1[1]} + We\,\eta_t^{2[1]}\Big|_{(10-i\&iii)} = 0 \tag{22-ii}$$

Where $v_{1t}, v_{1r}, v_{1rr}, v_{3t}, v_{3r}$, etc, .are partial derivatives.

Where $X^{[1]}$, $X^{[2]}$ and ($\eta_t^{1[1]}, \eta_r^{1[1]}, \eta_{rr}^{1[2]}, \eta_t^{2[1]}, \eta_r^{2[1]}$) are expressed in the relations (20 and 21). In the above equations, the unknown functions ϕ, ξ, η^1 and η^2 are independent for the differentials of v_1 and v_3. Thus separating w. r. to the differentials of v_1 and v_3 and powers of the differentials of v_1 and v_3 shows to the two simplified over resolved systems of PDE's and after solving these two over determined systems of linear PDE's, the values of the unidentified functions ϕ, ξ, η^1 and η^2 are is given by

$$\phi = c_1, \qquad \xi = 0, \qquad \eta^1 = c_2 v_1 + h_1(t,r) \text{ and } \qquad \eta^3 = c_2 v_3 + h_2(t,r) \tag{23}$$

Here $h_1(t,r)$ & $h_2(t,r)$ are arbitrary functions of *r* of the following partial differential equations.

$$\text{Re}\frac{\partial h_1}{\partial t} = \mu_2\frac{\partial^2 h_1}{\partial r^2} + \frac{\mu_2}{r}\frac{\partial h_1}{\partial r} + \frac{\partial h_2}{\partial r} + \frac{h_2}{r} - \frac{1}{Da}h_1 \quad \text{(a) and}\ We\frac{\partial h_2}{\partial t} = \mu_1\frac{\partial h_1}{\partial r} - h_2 \ \text{(b)} \tag{24}$$

In (23), c_1 and c_2 are constants of integration and symmetry Lie algebra of system of PDEs (10-i and10-iii) is two-dimensional and infinite and is generated by

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = v_1\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_3} \ \ \&\ X_m = h_1(t,r)\frac{\partial}{\partial v_1} + h_2(t,r)\frac{\partial}{\partial v_3} \tag{25}$$

Where *m* is any natural number

3.2.2 Invariant solution corresponding to X_1 - α X_2

The form of invariant result related in the generator $X = X_1 - \alpha X_2$ is given as

$$v_1(t,r) = e^{-\alpha t}\psi_1(r) \qquad \& \qquad v_3(t,r) = e^{-\alpha t}\psi_3(r) \tag{26}$$

The insertion of (26) into PDEs (10-i &10-iii) gives the reduced system of ordinary differential equations

$$\mu_2\psi_1''(r) + \frac{\mu_2}{r}\psi_1'(r) + \psi_3'(r) + \frac{1}{r}\psi_3(r) + (\alpha\,\text{Re} - \frac{1}{Da})\psi_1(r) = 0 \quad (i) \qquad \psi_3(r) = \frac{\mu_1}{1-\alpha We}\psi_1'(r) \quad (ii) \tag{27}$$

If we set the value of $\psi_3(r)$ from (27-ii) into (27-i), we have,

$$\psi_1''(r) + \frac{1}{r}\psi_1'(r) + \lambda^2\psi_1(r) = 0 \qquad \text{Where } \lambda^2 = \frac{(\alpha\,\text{Re} - \frac{1}{Da})(1-\alpha We)}{(1-\alpha\,\mu_2 We)} \tag{28}$$

Through Equation (11- i and ii), the corresponding boundary conditions are $\psi_1(1) = 0$ and $\psi_1'(0) = 0$ (29)

The above ordinary differential equation (28) is the Bessel's differential equation whose order is zero and similarly equation have been solved as in section 3.1 and then the general solution of Bessel's differential equation is

$$\psi_1(r) = c_1\, J_0(\lambda\, r) + \; c_2\, Y_0(\lambda\, r) \tag{30}$$

Where $J_0(\lambda r)$ and $Y_0(\lambda\, r)$ are Bessel function of order zero of first and second kind respectively. Hence $Y_0(\lambda r) \to -\infty$ when $r = 0$, so it is neglected, Therefore, the result is in one bounded solution is given as

$$\psi_1(r) = c_1\, J_0(\lambda\, r)$$

Now applying the boundary conditions from (34), then $\psi_1'(0) = 0$ is identically satisfied and

$\psi_1(1) = c_1\, J_0(\lambda) = 0 \quad \Rightarrow c_1 \neq 0$, So $J_0(\lambda) = 0$,

This has infinite number of roots λ_n $(n = 1, 2, 3, ..., \infty)$. Hence applying the superposition principle, we obtain the solution

$$\psi_1(r) = \sum_{n=1}^{\infty} c_n\, J_0(\lambda_n\, r) \tag{31-i}$$

and Substitute this value of $\psi_1(r)$ in equation (43-ii), then $\psi_3(r)$ is obtained i-e

$$\psi_3(r) = \sum_{n=1}^{\infty} \frac{-\mu_1\, \lambda_n\, c_n}{(1-\alpha\, We)}\, J_1(\lambda_n\, r) \tag{31-ii}$$

As $\lambda_n^2 = \dfrac{(\alpha\,\mathrm{Re} - \frac{1}{Da})(1-\alpha\, We)}{(1-\alpha\,\mu_2\, We)} \Rightarrow \alpha = \dfrac{1}{2}\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\;\mathrm{Re}}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\;\mathrm{Re}}\right)^2 - \frac{4(Da\;\lambda_n^2+1)}{4(Da\;\lambda_n^2+1)}}$

As solutions are obtained after combined two equations and have same boundary positions, so for the function of time, consider

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right)^2 - \frac{4(Da\,\lambda_n^2+1)}{Da\,\mathrm{Re}\,We}} \tag{32-i}$$

$$\alpha_2 = \frac{1}{2}\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right) - \frac{1}{2}\sqrt{\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right)^2 - \frac{4(Da\,\lambda_n^2+1)}{Da\,\mathrm{Re}\,We}} \tag{32-ii}$$

Therefore, equation (26) develops into as under:

$$v_1(t,r) = \sum_{n=1}^{\infty}\left(c_{n1}\, e^{-\alpha_1 t} + c_{n2} e^{-\alpha_2 t}\right) J_0(\lambda_n\, r) \text{ and } v_3(t,r) = \sum_{n=1}^{\infty} -\mu_1\,\lambda_n \left(\frac{c_{n1}\, e^{-\alpha_1 t}}{(1-\alpha_1 We)} + \frac{c_{n2}\, e^{-\alpha_2 t}}{(1-\alpha_2 We)}\right) J_1(\lambda_n\, r) \tag{33}$$

For the constants, applying the initial conditions (12-i) and (12-iii), so we obtain

$$v_1(0,r) = \sum_{n=1}^{\infty}\left(c_{n1} + c_{n2}\right) J_0(\lambda_n\, r) = -Da\left(1 - \frac{J_0(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)$$

$$\Rightarrow\; c_{n1} + c_{n2} = \frac{\int_0^1 -Da\left(1 - \frac{J_0(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) r\, J_0(\lambda_n\; r)\; dr}{\int_0^1 r\, J_0^{\,2}(\lambda_n\, r)\; dr} = \frac{-2\,Da}{\lambda_n\,(Da\,\lambda_n^2+1)\, J_1(\lambda_n)} \tag{34-i}$$

and similarly for the condition (12-iii), we have

$$\frac{c_{n1}}{1-\alpha_1 We)} + \frac{c_{n2}}{1-\alpha_2 We)} = \frac{-2\,Da}{\lambda_n\,(Da\,\lambda_n^2+1)\, J_1(\lambda_n)} \tag{34-ii}$$

After solving above equations (34-i and ii), we obtain \

$$c_{n1} = \frac{2\,Da\;(1-\alpha_1 We)\,\alpha_2}{(\alpha_1-\alpha_2)\,\lambda_n\,(Da\;\lambda_n^2+1)\, J_1(\lambda_n)} \quad \text{and} \quad c_{n2} = \frac{-2\,Da\;(1-\alpha_2 We)\,\alpha_1}{(\alpha_1-\alpha_2)\,\lambda_n\,(Da\;\lambda_n^2+1)\, J_1(\lambda_n)}$$

Substitute the values c_{n1} and c_{n2} in the relation (38), we get

$$v_1(t,r)=\sum_{n=1}^{\infty}\frac{2\,Da}{\lambda_n\,(Da\,\lambda_n^2-1)\,J_1(\lambda_n)}\left(\frac{(1-\alpha_1 We)\,\alpha_2\,e^{-\alpha_1 t}}{(\alpha_1-\alpha_2)}-\frac{(1-\alpha_2 We)\,\alpha_1\,e^{-\alpha_2 t}}{(\alpha_1-\alpha_2)}\right)J_0(\lambda_n r) \quad (35\text{-i})$$

$$v_3(t,r)=\sum_{n=1}^{\infty}\frac{-2\,\mu_1 Da}{(Da\,\lambda_n^2-1)\,J_1(\lambda_n)}\left(\frac{\alpha_2\,e^{-\alpha_1 t}}{(\alpha_1-\alpha_2)}-\frac{\alpha_1\,e^{-\alpha_2 t}}{(\alpha_1-\alpha_2)}\right)J_1(\lambda_n r) \quad (35\text{-ii})$$

3.2.2 Result of PDE (10-ii)

Now we take equation (10-ii), we have $We\dfrac{\partial v_2}{\partial t}+v_2=2We\{(v_3+\varphi_3(r))\dfrac{\partial v_1}{\partial r}+\varphi_1'(r)\,v_3\}$

According to the equation (35-i and ii), Solution of this equation is agreed as

$$V_2(t,r)=\left(\sum_{n=1}^{\infty}\lambda_n(2We\,\mu_1)^{\frac{1}{2}}\left(\frac{c_{n1}^2\,e^{-2\alpha_1 t}}{(1-\alpha_1 We)(1-2\alpha_1 We)}+\frac{c_{n2}^2\,e^{-2\alpha_2 t}}{(1-\alpha_2 We)(1-2\alpha_2 We)}+\frac{c_{n1}\,c_{n2}(2-\alpha_1 We-\alpha_2 We)\,e^{-(\alpha_1+\alpha_2)t}}{(1-\alpha_1 We)(1-\alpha_2 We)(1-\alpha_1 We-\alpha_2 We)}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2 \quad (36)$$

$$-2We\,\varphi_3(r)\sum_{n=1}^{\infty}\lambda_n\left(\frac{c_{n1}(2-\alpha_1 We)\,e^{-\alpha_1 t}}{(1-\alpha_1 We)^2}+\frac{c_{n2}(2-\alpha_2 We)\,e^{-\alpha_2 t}}{(1-\alpha_2 We)^2}\right)J_1(\lambda_n r)+e^{-\frac{t}{We}}\,\psi_2(r)$$

As $\varphi_3(r)=i\,\mu_1\sqrt{Da}\left(\dfrac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right)$

Consider $J_1(\frac{i}{\sqrt{Da}}r)=\sum_{n=1}^{\infty}c_n\,J_1(\lambda_n r)\;\Rightarrow\; c_n=\dfrac{2i\sqrt{Da}\;J_0(\frac{i}{\sqrt{Da}})}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}$ When $J_0(\lambda_n)=0$

So $J_1(\dfrac{i\,r}{\sqrt{Da}})=\sum_{n=1}^{\infty}\dfrac{2i\sqrt{Da}\;J_0(\frac{i}{\sqrt{Da}})}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}J_1(\lambda_n r)$

After putting the values of c_{n1}, c_{n2} and $\varphi_3(r)$, so the equation (36) turns into

$$v_2(t,r)=\left(\sum_{n=1}^{\infty}\frac{Da\,(8\,We\;\mu_1)^{\frac{1}{2}}}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\left(\begin{array}{l}\dfrac{\alpha_2^2(1-\alpha_1 We)\,e^{-2\alpha_1 t}}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_1 We)}+\dfrac{\alpha_1^2(1-\alpha_2 We)\,e^{-2\alpha_2 t}}{(\alpha_1-\alpha_2)^2(1-2\alpha_2 We)}+\dfrac{\alpha_2\,(2-\alpha_1 We)\,e^{-\alpha_1 t}}{(\alpha_1-\alpha_2)(1-\alpha_1 We)}\\-\dfrac{\alpha_1\,(2-\alpha_2 We)\,e^{-\alpha_2 t}}{(\alpha_1-\alpha_2)(1-\alpha_2 We)}-\dfrac{\alpha_1\alpha_2\,(2-\alpha_1 We-\alpha_2 We)\,e^{-(\alpha_1+\alpha_2)t}}{(\alpha_1-\alpha_2)^2(1-\alpha_1 We-\alpha_2 We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2+e^{\frac{-t}{We}}\,\psi_2(r)$$

Now applying the initial condition (12-ii), then we obtain,

$$\psi_2(r)=2We\,\mu_1\,Da\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)^2-\left(\sum_{n=1}^{\infty}\frac{Da\;(8\,We\;\mu_1)^{\frac{1}{2}}}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\left(\begin{array}{l}\dfrac{\alpha_2^2(1-\alpha_1 We)}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_1 We)}+\dfrac{\alpha_1^2(1-\alpha_2 We)}{(\alpha_1-\alpha_2)^2(1-2\alpha_2 We)}+\dfrac{\alpha_2\,(2-\alpha_1 We)}{(\alpha_1-\alpha_2)(1-\alpha_1 We)}\\-\dfrac{\alpha_1\,(2-\alpha_2 We)}{(\alpha_1-\alpha_2)(1-\alpha_2 We)}-\dfrac{\alpha_1\alpha_2(2-\alpha_1 We-\alpha_2 We)}{(\alpha_1-\alpha_2)^2(1-\alpha_1 We-\alpha_2 We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2 \quad (37)$$

Hence the final result of the system (4 to 6) admit the following solutions

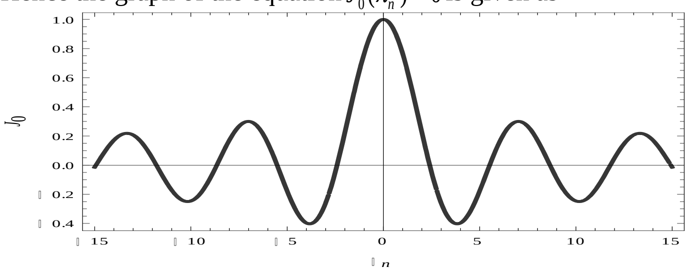
$$u(t,r)=\sum_{n=1}^{\infty}\frac{2\,Da}{\lambda_n\,(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\left(\frac{(1-\alpha_1 We)\,\alpha_2\,e^{-\alpha_1 t}}{(\alpha_1-\alpha_2)}-\frac{(1-\alpha_2 We)\,\alpha_1\,e^{-\alpha_2 t}}{(\alpha_1-\alpha_2)}\right)J_0(\lambda_n r)+Da\left(1-\frac{J_0(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) \quad (38\text{-i})$$

$$\tau_{11}(t,r)=\left(\sum_{n=1}^{\infty}\frac{Da\,(8We\ \mu_1)^{\frac{1}{2}}}{(Da\,\lambda_n^2+1)J_1(\lambda_n)}\left(\begin{array}{l}\frac{\alpha_2^2(1-\alpha_1 We)\,(e^{-2\alpha_1 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_1 We)}+\frac{\alpha_1^2(1-\alpha_2 We)\,(e^{-2\alpha_2 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_2 We)}+\frac{\alpha_2\,(2-\alpha_1 We)(e^{-\alpha_1 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)\,(1-\alpha_1 We)}\\-\frac{\alpha_1\,(2-\alpha_2 We)(e^{-\alpha_2 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)(1-\alpha_2 We)}-\frac{\alpha_1\alpha_2(2-\alpha_1 We-\alpha_2 We)\,(e^{-(\alpha_1+\alpha_2)t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2(1-\alpha_1 We-\alpha_2 We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2 \tag{38-ii}$$

$$+\ 2\,We\,\mu_1\,Da\left(e^{-\frac{t}{We}}-1\right)\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)$$

$$\tau_{12}\ (t,r)=\sum_{n=1}^{\infty}\frac{-2\,\mu_1 Da}{(Da\,\lambda_n^{\,2}+1)\,J_1(\lambda_n)}\left(\frac{\alpha_2\,e^{-\alpha_1 t}}{(\alpha_1-\alpha_2)}-\frac{\alpha_1\,e^{-\alpha_2 t}}{(\alpha_1-\alpha_2)}\right)J_1(\lambda_n r)+i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) \tag{38-iii}$$

Hence the graph of the equation $J_0(\lambda_n)=0$ is given as



Figure–1: Graph of the equation $J_0(\lambda_n)=0$

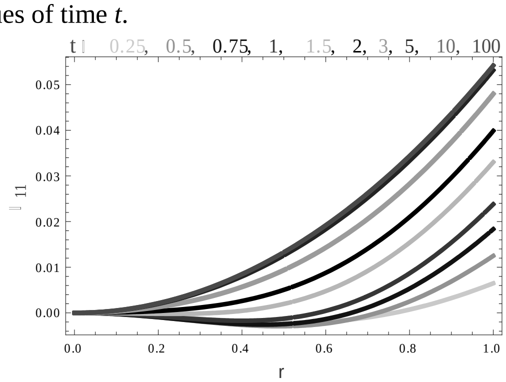
This figure-1 shows the different values of

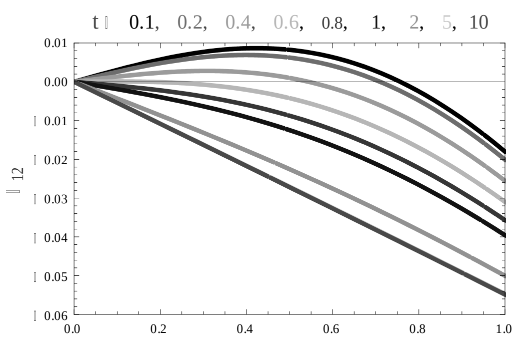
$\lambda_n=\pm 2.40482555, \pm 5.52007812, \pm 8.6537279, \pm 11.7915344\ \ldots\ldots\ldots\ldots, \infty$, which satisfy the equation $J_0(\lambda_n)=0$, so choose the one value $\lambda=2.40482555$ for the graph.

4. ANALYSIS OF VISCOELASTIC STRESSES

4.1 Analytical solutions of normal and shear stresses and its graphs. .

The results of analytical solutions of normal stress component $\tau_{11}(t,r)$ and shear stress component $\tau_{12}(t,r)$ are plotted in figures 2–3 for several parameters with $\mathrm{Re}=1,\ We=1,\ Da=10,\ \mu_1=\frac{1}{9},\ \mu_2=\frac{8}{9}$ and at different values of time *t*.





Fig–2: Result of normal stress component τ_{11} of (38-ii)

Fig–3: Result of shear stress component τ_{12} of (38-iii)

The result of relation (38-ii) and (38-iii) demonstrated analytical solution of normal stress component τ_{11} and the shear stress component τ_{12} related with time dependent equations of the system (4 to 6) and the result of component of normal stress profile τ_{11} is presented into figure–2 and show with the aim of the first component τ_{11}

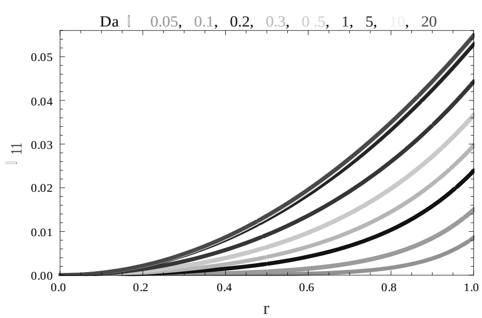
expands with rising time and get to a higher state line at similar time rank into non–linear style and reaches at steady state at a maximum value of 0.0542. Whilst, the behavior the shear stress component τ_{12} is illustrated in figure–3 and shear stress τ_{12} clearly indicate the linear movement in reduces by means of expand in time since it shall exist. There is no other alters in shear–stress since flow of fluid approaches at a minimum value which is equal to - 0.0555 and flow become steady state.

4.2 Study state solution of normal and shear stresses and its Graphs.

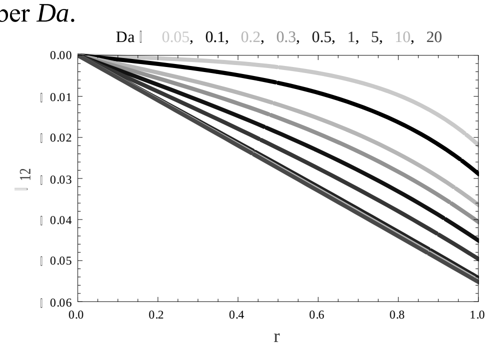
The invariant solution related with X_1 is the steady-state solution which is already found in the section: 3.1 in the relation (16).and steady-state solutions of normal stress component $\tau_{11}(r)$ and shear stress component $\tau_{12}(r)$ is given as

$$\tau_{11}(r)=\varphi_2(r)=-2We\,\mu_1\,Da\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right)^2 \quad \text{(39-i)} \qquad \tau_{12}(r)=\varphi_3(r)=i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right) \quad \text{(39-ii)}$$

The results of steady state solutions are plotted in figures 4 & 5 with $\mu_1=\frac{1}{9}$, $We=1$ and at different values of Darcy's number *Da*.



Fig–4: Steady state solution of normal stress component τ_{11}



Fig–5: Steady–state solution of shear stress component τ_{12}

The steady state results of components of normal stress and shear stress show in **(Fig 4and 5)** respectively. As the **Fig** 4 illustrate in order to the steady normal stress component τ_{11} at high Darcy's number (*Da*) flow behave like without porous media whilst, as Darcy's number (porosity) of porous media decreases and flow resistant of the fluid expands and here component τ_{11} reduces within the steady condition and as figure–5 demonstrates to facilitate in the steady state, when pipe flow containing small values of *Da*, then component of shear stress τ_{12} contains big values so that if permeability reduces, then component τ_{12} enlarges in the steady situation and eventually there is no flow when Darcy's number approaches to vanishing value.

5. CONCLUSIONS

The main purpose of this research paper was to find the analytical solutions of viscoelastic fluid flow in pipes filled with porous medium in conjunction of the constant viscosity Oldroyd–B constitutive model by using Lie group analysis and most important analysis of this paper is to discover the exact results of normal and shear stress components. for non homogenous PDE, firstly steady state solution have been found and changed the original PDE's system in to new PDE's system in the new dependent variables subject to suitable boundary conditions and new initial conditions. Lie-point symmetries have been achieved by applying symmetry method and accepted to arrive at the result of the problems and can supply a number of valuable insights interested in the solutions structure and sometimes can be helpful to find the exacting solutions in a lot of cases. We hope that the results may be accommodating for other human resources in the field.

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