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Absolute Stability for a Fractional Numerical Algorithm

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Abstract: In this paper, analysis of absolute stability is presented for a Fractional Euler Method found in literature. Both interval and region for absolute stability shown graphically have been found equivalent to absolute stability for classical Euler Method in case of integer-order ordinary differential equations.

Keywords: Absolute Stability, Fractional Euler Method, Region of Stability.

1. INTRODUCTION

The birth of Fractional Calculus took place on the same day when Classical (Integer-Order) Calculus was born but its applications in diverse fields of Science and Engineering have been acclaimed some three decades ago (Podlubny, 1999). Detailed discussion in the work of (Dalir and Bashour, 2010) shows that Fractional Differential Equations are found in fields like Control Theory, Neuroscience, Viscoelasticity of materials, Fluid and Quantum Mechanics, Brownian Motion; to name a few. One can go through the work of (Debnath, 2003) to know more about areas of applications for Fractional Differential Equations. Most of these equations are nonlinear in nature and cannot be solved in terms of elementary and special mathematical functions leading to development of new numerical algorithms to get the acceptable solutions (Li and Zeng, 2015). Two new methods called the decomposition method and the variational iteration method are relatively common approaches to get an analytical approximate solution to linear and nonlinear fractional differential equations (Diethelm. 1997). Many researchers have investigated possibilities for fractional differential equations to have unique solution (Yakar and Koksal, 2012). Although, there are many varied definitions for fractional derivatives but two most commonly used are of Riemann-Liouville and Caputo:

Definition 1. Suppose that $\alpha > 0$, $t > 0$, $\alpha, t \in \Re$. Then

$$ {}_{RL}D_0^{\alpha} y(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t \dfrac{y(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n \in N \\ \dfrac{d^n}{dt^n} y(t), & \alpha = n \in N \end{cases} $$

is called the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order α.

Definition 2: Suppose that $\alpha > 0$, $t > 0$, $\alpha, t \in \Re$. Then

$$ {}_{C}D_0^{\alpha} y(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{y^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n \in N \\ \dfrac{d^n}{dt^n} y(t), & \alpha = n \in N \end{cases} $$

is called the Caputo fractional derivative or the Caputo fractional differential operator of order α. Riemann and Caputo fractional derivatives are related with each other by the following relationship:

$$ {}_{C}D_0^{\alpha} y(t) = {}_{RL}D_0^{\alpha} y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} $$

There are numerous papers where one can find discretization for Riemann-Liouville and Caputo derivatives in order to develop numerical algorithms for

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finding solution of fractional differential equations such as in (Garrappa. 2009). (Tong. *et al*, 2013) have proposed an Euler Method with its convergence for fractional differential equations in Riemann-Liouville sense whereas (Odibat and Momani, 2008) have devised a Fractional Euler Method in Caputo sense which is the main concern of the present paper since analysis regrading stability for the proposed Fractional Euler Method is not found in the relevant literature.

2. MATERIAL AND METHODS

Following definition is important to know before starting discussion regarding absolute stability.

Definition 3. A numerical algorithm is said to be absolutely stable for a given $z \in C$ if all the roots of stability function $\phi(z)$ lie within the unit circle and a region R_A of the complex plane is said to be a region of absolute stability if the algorithm is stable for all z in R_A. Development of the method proposed by (Odibat and Momani, 2008) is as follows:

Consider the initial value problem of fractional order

$$D^{\alpha} y(t) = f\left(t, y(t)\right),\; y(0) = y_0,\;\; 0 < \alpha \le 1,\; t > 0.$$

Use of following Generalized Taylor's Series is very important in the development of the Fractional Euler Method as stated by (Odibat and Momani, 2008).

Suppose that $D^{k\alpha} y(t) \in C(0, a]$ for $k = 0, 1, \cdots, n+1$, where $0 < \alpha \le 1$. Then we have

$$y(t) = \sum_{i=0}^{n} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}\right) y(0+) + \frac{\left(D^{(n+1)\alpha}\right) y(\xi)}{\Gamma\left((n+1)\alpha+1\right)} t^{(n+1)\alpha},$$

with $0 \le \xi \le t,\; \forall\;\; t \in (0, a].$

Using this generalized Taylor's formula, following Fractional Euler Method was derived without giving further analysis of the method:

$$y_{n+1} = y_n + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\left(t_n, y_n\right);\;\; n = 0, 1, 2, \cdots \quad (1)$$

where $\Gamma(\cdot)$ is an Eulerian-Gamma Function. This algorithm clearly retains Classical Explicit Euler Methodfor $\alpha = 1$.

2.1 Absolute Stability of Fractional Euler Method

Analysis of absolute stability is an important feature in the study of numerical approximation of fractional differential equations. A linear model problem for fractional differential equations is given as

$$D^{\alpha} y(t) = \lambda y(t),\;\; \lambda \in C,\; 0 < \alpha \le 1$$

whose exact solution is expressed in terms of the one-parameter Mittag-Leffler function:

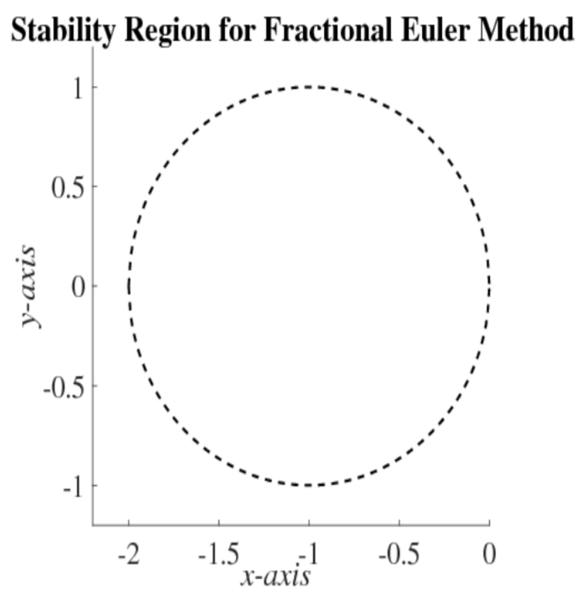
$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$ as

$$y(t) = E_{\alpha}\left(\lambda\left(t - t_0\right)^{\alpha}\right) y_0 \cdot$$

Applying the Fractional Euler Method, we get

$$y_n = y_0 \left(1 + \frac{h^{\alpha}\lambda}{\Gamma(\alpha+1)}\right)^n$$

Letting $z = \dfrac{h^{\alpha}\lambda}{\Gamma(\alpha+1)}$, we get the stability function as $\phi(z) = 1 + z$ and in order to be absolutely stable, Fractional Euler Method requires $\left|\phi(z)\right| < 1$ which is an open unit disk shown below:



3. RESULTS AND DISCUSSION

The region inside the unit disk is the region of absolute stability for Fractional Euler Method whereas interval of stability from the graph is observed to be $(-2,0)$. It is to be noted that both interval and region of absolute stability for Fractional Euler Method is exactly the same for Classical Explicit Euler Method known to us from Classical Calculus.

4. CONCLUSION

This work was mainly concerned with analysis of absolute stability of a Fractional Euler Method found in literature. Interval and region of absolute stability were shown graphically which were also observed to be equivalent to stability analysis of Classical Explicit Euler Method used to solve integer-order initial value problems in Calculus known to us.

REFERENCES:

Debnath L, (2003), "Recent Applications of Fractional Calculus to Science and Engineering",IJMMS, Hindawi Publishing Corp, (54) 3413-3442.

Diethelm, K. (1997)"An algorithm for the numerical solution of differential equations of fractional order," Electronic transactions on numerical analysis, vol. 5, no. 1, 1–6.

Garrappa R, (2009), "On Some Explicit Adams Multistep Methods for Fractional Differential Equations", Journal of Applied and Computational Mathematics, (229), 392-399.

Li C and F, Zeng (2015), "Numerical Methods for Fractional Calculus", CRC Press Taylor & Francis Group.

Mehdi D. and M. and Bashour (2010), "Applications of Fractional Calculus", Applied Mathematical Sciences, 21(4), 1021-1032.

Odibat. Z. M and S. bMomani. (2008), "An Algorithm for the Numerical Solution of Differential Equations of Fractional Order", J. Appl. Math & Informatics, 26(1), 15-27.

Podlubny, I. (1999), "Fractional Differential Equations", Vol. 198, Academic Press, USA.

Tong P, and Y Feng (2013), "Euler's Method for Fractional Differential Equations", WSEAS Transactions on Mathematics, 12(12), 1146-1153.

Yakar A. and M. E. Koksal, (2012), "Existence results for solutions of nonlinear fractional differential equations," in Abstract and Applied Analysis, vol. 2012, Hindawi Publishing Corporation.