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Study for Darcy-Brinkman model connected Oldroyd–B Constitutive Model related with Solutions of Velocity of Viscoelastic Fluid Flow in a circular Pipe filled with Porous Media

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Abstract: The basic purpose of this research paper is to investigate the solution of velocity of viscoelastic fluid flow with Porous media in a circular pipe to use the Lei- group technique to the system of PDE's comprising the continuity, momentum and constitutive; equations, under appropriate initial and boundary conditions. The associated problem is organized by using the Darcy-Brinkman model connected with Oldroyd–B constitutive model. The investigation is arranged through analytical solutions of the PDE's system connected with initial and boundary value problem. The analytical solution is established under the symmetries for the system through Lie group method.

Keywords: Viscoelastic fluid flow through Porous media in circular pipe, Darcy-Brinkman model, Oldroyd–B constitutive model, Lie group technique, investigation of Velocity.

1 INTRODUCTION

Fluid Mechanics is a subject of science which deals with basic concepts and principles in hydrostatics, hydro kinematics and hydrodynamics and their application in solving fluid flow problems. In the fluid dynamics, Newtonian and non-Newtonian fluids are expressively concerned great to interest in the literature. The results of Newtonian and Non–Newtonian fluid flow in problems classically depend on different properties of the flow of fluid, for example, velocity, temperature, pressure and density, as functions related with space and time. Fluid flow connected with porous media proposes a systematic structure that lies under realistic rules and that holds experiential and semi-experiential laws which are obtained from of flow and used to find the solution of practical problems. Flows of Newtonian and non-Newtonian fluids related with some essential investigation are prepared by way of Ariel *et al.* (2006), Abel-Malek *et al* (2002), Chen *et al.* (2006), Bird *et al.* (1987), Fetecau and Fetecau (2005. 2006), Rajagopal and Gupta (1984), Rajagopal and Na (1985), and Wafo-(2005).

The research is to obtain a model that is as easy as likely, relating the minimum number of variables and parameters, and until now containing the facility to find out the viscoelastic activities in compound fluid flows investigated by Hulsen (1990) and Keunings (2003). A general feature of viscoelastic fluid is stress rest later than an unexpected shearing displacement where stress exceeds to a maximum after that begins lessening exponentially and ultimately reconciles to a value of steady state. A common consent has appeared that the flow with porous media related with viscoelastic fluids elastic effects should come up, even if their accurate nature is unidentified or contentious. Viscoelastic effects in porous media can be imperative insure cases. Whilst in these, the genuine pressure gradient will go beyond the simply viscous gradient further than a serious flow rate, as looked at by some researchers.

Oldroyd-B model is the nonlinear viscoelastic model and is a second simplest model and it seems that the most well-liked in fluid flow viscoelastic modeling. Here viscoelastic behaviour will be modelled by the Oldroyd-B (Oldroyd 1958) and Phan-thien/Tanner (PTT) 1977) differential constitutive models and simulation developed by van Os; Phillips. (2004), Larson (1999 and Taha 2010) due to their large applications. The result of problems concerned with realistic fluid flow solved with Lie Group techniques has obtained rising concentration during current years. Lie-Group theory of ODE's and PDE's as a scientific branch created from efforts of the exceptional mathematician Lie of the nineteenth century (1842–1899) and developed by Bluman and Kumei (1989), Olver, (1986), Ibragimov, (1999) and others.and since then it has existed the major constituent part of his most significant creation of the continuous groups theory. For PDE's, Lie point symmetries allow the reduction of the number of independent variables expanded by Abdel-Malek, Badran and Hassan (2002), Basov's. (2004), Moran, and Gaggioli, (1968) and others.

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This paper is associated with the investigative solutions of viscoelastic Fluid flow in a pipe using Oldroyd–B constitutive model. Analytical solutions are obtained in the way using symmetries of the system through Lie Group method and results are given and the physical interpretations of the solutions are prepared through graphs, these graphs are discussed. Finally the conclusions of this paper are given.

Section 2 is related with the problem formulation. Section 3 connected with viscoelastic fluid flow solution in circular pipes filled with porous media; section 3.1 attached according to non-homogeneous equation (4-i), section 3.2 concerns with symmetries of the PDE's (13-i & iii), Section 3.3 associated with invariant solution of PDE's (13-i & iii) corresponding to $X_1 - \alpha X_2$, Section 3.3,1 related with solution of PDE's (13-ii). Section 4 is connected with analysis of Velocity. Hence graph of invariant solution of velocity of equation (36-i) are given and discussed in section 4.1and graph of steady-state solution of the velocity is discussed in section 4.2. Also section 5 connected with conclusions of these problems.

2 PROBLEM FORMULATIONS

Suppose viscoelastic fluid flow through porous media which is unsteady incompressible laminar flow apprehended in a circular pipe drenched in radial direction. A system of cylindrical polar coordinate is applied with radius-axis vertically upward. The most important equations system of flow contains of the conservation of both mass and momentum transport related with the Oldroyd–B constitutive model. In the absence of body force adopting Darcy-Brinkman model transfers a system of equations is used. The viscoelastic fluid flow with porous medium is supposed to be homogeneous and isotropic. As the flow in pipe is supposed to exist unidirectional expressed within only axial velocity as a function of redial direction along hydro dynamically entirely expanded which velocity does not hinge on the axial route of the pipe and the pressure gradient is supposed to live constant. For unidirectional flow velocity field is given as $\bar{u} = (u(r,t),0,0)$; wherever the above meaning of velocity mechanically satisfies the incompressibility state. The continuity equation, generalized Darcy–Brinkman model has been employed for the momentum equation through porous media and the Oldroyd–B equation define the stresses of viscoelastic in the fluid flow in vectorial form can be written as under:

$$\nabla.\bar{u} = 0 \quad (1) \quad \text{and} \quad \frac{\rho}{\varepsilon}\frac{\partial \bar{u}}{\partial t} = \frac{1}{r}\nabla([\frac{\mu_2}{\varepsilon} r \underline{\underline{d}}] + \tau) - \nabla p - \rho\,\bar{u}.\nabla\bar{u} - \frac{\mu}{K}\bar{u} \quad (2)$$

The Oldroyd–B constitutive equation describes the viscoelastic stresses in the flow can be expressed as below:

$$\lambda \frac{\partial \tau}{\partial t} = [2\mu_1 \underline{\underline{d}}] - \tau - \lambda\{\bar{u}.\nabla\tau - \nabla\bar{u}.\tau - (\nabla\bar{u})^T.\tau\} \quad (3)$$

In the above equations, $\bar{u}$ is the velocity vector field of flow, τ is the extra stress tensor, $\underline{\underline{d}}$ is the rate–of–strain tensor, ∇ is the spatial differential operator, p is the isotropic fluid pressure (per unit density) and t is the time. The μ_1 and μ_2 are respectively the viscoelastic solute and Newtonian solvent viscosities, fluid density is denoted by ρ, whereas λ is the relaxation time of the viscoelastic fluid and K is the intrinsic permeability of the porous medium,. Total viscosity μ of the viscoelastic flow is $\mu = \mu_1 + \mu_2$ and is taken constant and ε is porosity of porous media.
The equations are derived which govern the unsteady unidirectional fluid flow through porous of viscoelastic fluid media adopting Oldroyd–B constitutive model. For unidirectional flow the velocity field is $\bar{u} = (u(r,t),0,0)$; here the description of velocity automatically gives pleasure to the incompressibility state. The derivation of such equations by employing the momentum transport equation of viscoelastic fluid and Oldroyd–B constitutive equations assuming constant pressure gradient and may be expressed in the absence of body force, the governing system of equations is written in the dimensionless form as under

$$\mathrm{Re}\frac{\partial u}{\partial t} = 1 + \mu_2\frac{\partial^2 u}{\partial r^2} + \frac{\mu_2}{r}\frac{\partial u}{\partial r} + \frac{\partial \tau_{12}}{\partial r} + \frac{\tau_{12}}{r} - \frac{1}{Da}u \quad \text{(i)}$$

$$We\frac{\partial \tau_{11}}{\partial t} = 2We\,\tau_{12}\frac{\partial u}{\partial r} - \tau_{11} \quad \text{(ii)} \quad \& \quad We\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial u}{\partial r} - \tau_{12} \quad \text{(iii)} \quad (4)$$

Where u(r, t) and $\tau\,(r,t)$ are dimensionless velocity in the axial direction and dimensionless stress tensor in axial, shear and radial direction, r is radial coordinates, t is the time using for non-dimensional. Where the non-dimensional Reynolds number (R*e*), Weissenberg number (*We*) and Darcy's number (*Da*) are identified as $\mathrm{Re} = \frac{R\,\rho\,Vc}{\mu}$, $We = \frac{\lambda\,Vc}{R}$, $Da = \frac{K}{\varepsilon\,R^2}$ and $\mu_1 + \mu_3 = 1.$

As K is the adapted permeability concern with the porous medium using for non-dimensional. As R is a radius of the pipe and Vc is used for the feature velocity supposed since reference redial velocity $Vc = \frac{\varepsilon\,R^2\left(-\frac{\partial p}{\partial z}\right)}{\mu}$

Initial and boundary conditions for completing the well posed problem are taken as

$u\,(t,1) = 0,$ and $\frac{\partial u}{\partial t}(t,0) = 0$ When t > 0 (5)

and initial conditions are taken as $u(0,r) = \tau_{11}(0,r) = \tau_{12}(0,r) = 0$ When $0 < r < 1$ (6)\

1. Viscoelastic fluid flow Solutions in Circular Pipes filled with Porous Media

The PDE's system (4) subject to initial and boundary conditions (5 & 6) is solved by finding the firstly steady state solution according to non-homogenous equation (4-i)

3.1 **According to Non homogenous Equation (4-i)**

A few problems involving non–homogeneous equations or boundary conditions can be determined by means of the change of dependent variable, $u = v + \psi$

The basic idea to determine ψ , a function of one variable, in such a manner that v, a function of two variables, is made to satisfy a homogeneous PDE or homogeneous conditions of boundary. For the non–homogeneous governing Equation (4-i), a change of dependent variables and to find the steady state solution, hence, consider

$u(t,\,r) = v_1(t,\,r) + \psi_1(r),\;\; \tau_{11}(t,\,r) = v_2(t,\,r) + \psi_2(r)$ and $\tau_{12}(t,\,r) = v_3(t,\,r) + \psi_3(r)$ (7)

Substituting above values in Equation (4), gives the two systems of equations which are

$\mu_2\,\psi_1''(r) + \frac{\mu_{2r}}{}\psi_1'(r) + \psi'_3(r) + \frac{\psi_3(r)}{r} - \frac{1}{Da}\psi_1(r) + 1 = 0$ (i) $\psi_2(r) = 2We\;\psi_3(r)\;\psi_1'(r)$ (ii)

and $\psi_3(r) = \mu_1\;\psi_1'(r)$ (iii) (8)

Subject to boundary conditions $\psi_1(1) = 0$ and $\psi_1'(0) = 0$ (9)

For solving the system of equations (8), put the $\psi_3(r)$ from (8-iii) in to (8-i), it gives,

$\psi_1''(r) + \frac{1}{r}\psi_1'(r) - \frac{1}{Da}\psi_1(r) + 1 = 0$ (10)

Integrating the above ODE by using power series solution and applying the boundary conditions, so obtained the result as under

$$\psi_1(r) = Da\left(1 - \frac{j_0\left(\frac{i\,r}{\sqrt{Da}}\right)}{j_0\left(\frac{i}{\sqrt{Da}}\right)}\right) = Da\left(1 - \frac{\sum\limits_{n=0}^{\infty}(4Da)^{-n}\,(n!)^{-2}\,r^{2n}}{\sum\limits_{n=0}^{\infty}(4Da)^{-n}\,(n!)^{-2}}\right) \quad (11)$$

Here $J_0(\frac{i}{\sqrt{Da}}r)$ is Bessel function of order zero of first kind, respectively.

Substitute this value of $\psi_1(r)$ in equation (8-iii), then $\psi_3(r)$ is obtained. After substituting the values of $\psi_1'(r)$ and $\psi_3(r)$ in equation (8-ii), then $\psi_2(r)$ is obtained. Then the following solution of system of equations (8) is achieved

$$\psi_1(r) = Da\left(1 - \frac{J_0\left(\frac{i}{\sqrt{Da}}r\right)}{J_0\left(\frac{i}{\sqrt{Da}}\right)}\right),\quad \psi_2(r) = -2We\,\mu_1\,Da\left(\frac{J_1\left(\frac{i}{\sqrt{Da}}r\right)}{J_0\left(\frac{i}{\sqrt{Da}}\right)}\right)^2,\ \psi_3(r) = i\,\mu_1\sqrt{Da}\left(\frac{J_1\left(\frac{i}{\sqrt{Da}}r\right)}{J_0\left(\frac{i}{\sqrt{Da}}\right)}\right) \quad (12)$$

Thus to determine $v_1(t,r), v_2(t,r)\ \&\ v_3(t,r)$, the new boundary value problem is given as

$\mathrm{Re}\;v_{1t} = \mu_2\,v_{1rr} + \frac{\mu_2}{r}v_{1r} + v_{3r} + \frac{v_3}{r} - \frac{1}{Da}v_1$ (i)

$$We\ v_{2t} = 2\ We\ \{(v_3 + \psi_3(r))\, v_{1r} + \psi_1'(r) v_3\} - v_2 \quad \text{(ii)} \qquad We\ v_{3t} = \mu_1 v_{1r} - v_3 \quad \text{(iii)} \qquad (13)$$

Subject to initial and boundary conditions are,

$$v_1(0,r) = -\psi_1(r) \ \text{(i)} \quad v_2(0,r) = -\psi_2(r) \ \text{(ii)} \quad v_3(0,r) = -\psi_3(r) \ \text{(iii)} \quad 0 \le r \le 1 \qquad (14)$$

$$v_1(t,1) = 0, \quad \text{(i)} \qquad v_{1r}(0) = 0 \quad \text{(ii)} \qquad t > 0 \qquad (15)$$

Where $v_{1t} = \dfrac{\partial v_1}{\partial t}$, $v_{1r} = \dfrac{\partial v_1}{\partial r}$, v_{1rr}, v_{3t}, v_{3r}, etc, .are partial derivatives.

3.2 Symmetries Analysis of the PDE's (13-i & 13- iii)

Once symmetry Lie algebra of the differential equation is known, it can be used in the investigation of transformations that will reduce the equation to simpler form and it is powerful method in obtaining analytical solutions of differential equations. In this section, symmetry conditions and method for finding the Lie point symmetries of the equations (13-i &13-iii) (because derivatives of these equations are connected each other) are introduced. The operator.

$$X = \phi(t,r,v_1,v_3)\frac{\partial}{\partial t} + \xi(t,r,v_1,v_3)\frac{\partial}{\partial r} + \eta^1(t,r,v_1,v_3)\frac{\partial}{\partial v_1} + \eta^2(t,r,v_1,v_3)\frac{\partial}{\partial v_3} \qquad (16)$$

is the Lie point symmetry generator for governed system of partial differentials equations (13-i &13-iii) iffy,

$$X^{[2]}\left(\mu_2 v_{1rr} + \frac{\mu_2}{r} v_{1r} + v_{3r} + \frac{v_3}{r} - \tfrac{1}{Da} v_1 - \mathrm{Re}\ v_{1t}\right)\Big|_{(13-i\ \&\ iii)} = 0$$

$$X^{[1]}(We\ v_{3t} - \mu_1 v_{1r} + v_3)\Big|_{(13-i\,\&\,iii)} = 0$$

Where first and second extended infinitesimal generator of *X* are

As $X^{[1]} = X + \eta_t^{1[1]}\dfrac{\partial}{\partial u_t} + \eta_r^{1[1]}\dfrac{\partial}{\partial v_{1r}} + + \eta_t^{2[1]}\dfrac{\partial}{\partial v_{3t}} + \eta_r^{2[1]}\dfrac{\partial}{\partial v_{3r}}$, $\quad X^{[2]} = X^{[1]} + \eta_{rr}^{1[2]}\dfrac{\partial}{\partial v_{1rr}} + \ldots.$ (17)

In the operator *X*, according to Lie's theory, the unknown functions ϕ, ξ, η^1, η^2 are taken independent of the derivatives of the primitive variables v_1 and v_3 and established from the determining equations derived from the invariance condition.

In which $\eta_t^{1[1]}$, $\eta_r^{1[1]}$, $\eta_t^{2[1]}$, $\eta_r^{2[1]}$, $\eta_{rr}^{1[2]}$ are written by

$$\eta_r^{1[1]} = D_r\eta^1 - v_{1t}\, D_r\phi - v_{1r}\, D_r\xi; \qquad \eta_t^{1[1]} = D_t\eta^1 - v_{1t}\, D_t\phi - v_{1r}\, D_t\xi\ ;$$

$$\eta_{rr}^{1[2]} = D_r\eta_r^{1[1]} - v_{1tr}\, D_r\phi - v_{1rr}\, D_r\xi\ ;$$

$$\eta_t^{2[1]} = D_t\eta^2 - v_{3t}\, D_t\phi - v_{3r}\, D_t\xi\ ; \qquad \eta_r^{2[1]} = D_r\eta^2 - v_{3t}\, D_r\phi - v_{3r}\, D_r\xi\ ; \qquad (18)$$

Where D_t and D_r are the total derivative operators given as

$$D_t = \frac{\partial}{\partial t} + v_{1t}\frac{\partial}{\partial v_1} + v_{1tr}\frac{\partial}{\partial v_{1t}} + v_{1tt}\frac{\partial}{\partial v_{1t}} + v_{3t}\frac{\partial}{\partial v_3} + v_{3r}\frac{\partial}{\partial v_{3tr}} + v_{3tt}\frac{\partial}{\partial v_{3t}} + v_{3tr}\frac{\partial}{\partial v_{3r}} + \ldots\ldots\ldots,$$

$$D_r = \frac{\partial}{\partial r} + v_{1r}\frac{\partial}{\partial v_1} + v_{1tr}\frac{\partial}{\partial v_{1t}} + v_{1rr}\frac{\partial}{\partial v_{1r}} + v_{3r}\frac{\partial}{\partial v_3} + v_{3t}\frac{\partial}{\partial v_{3tr}} + v_{3rr}\frac{\partial}{\partial v_{3r}} + v_{3tr}\frac{\partial}{\partial v_{3t}} + \ldots\ldots.$$

As $X^{[2]}\left(\mu_2 v_{1rr} + \dfrac{\mu_2}{r} v_{1r} + v_{3r} + \dfrac{v_3}{r} - \tfrac{1}{Da} v_1 - \mathrm{Re}\ v_{1t}\right)\Big|_{(13-i\,\&\,iii)} = 0$

$$\Rightarrow\ -\frac{1}{r^2}(\mu_2 v_{1r} + v_3)\xi - \tfrac{1}{Da}\eta^1 + \frac{1}{r}\eta^2 - \mathrm{Re}\ \eta_t^{1[1]} + \frac{\mu_2}{r}\eta_r^{1[1]} + \eta_r^{2[1]} + \mu_2\,\eta_{rr}^{1[2]}\Big|_{(13-i\ \&\ iii)} = 0 \qquad (19)$$

$$X^{[1]}(We\ v_{3t} - \mu_1 v_{1r} + v_3)\Big|_{(13-i\,\&\,iii)} = 0\ \Rightarrow \eta^2 - \mu_1\eta_r^{1[1]} + We\ \eta_t^{2[1]}\Big|_{(13-i\,\&\,iii)} = 0 \qquad (20).$$

Where $X^{[1]}$, $X^{[2]}$ and ($\eta_t^{1[1]}, \eta_r^{1[1]}, \eta_{rr}^{1[2]}, \eta_t^{2[1]}, \eta_r^{2[1]}$) are expressed in the relations (18). In the above equations (19 and 2o), the unknown functions ϕ, ξ, η^1 and η^2 are independent for the differentials of v_1 and v_3. Thus equating and separating them by the derivatives of v_1 and v_3 and powers of the derivatives of v_1 and v_3 deals to the two simplified over resolved systems of PDE's and after solving these two over determined systems of linear PDE's gives rise to the values of the unidentified functions ϕ, ξ, η^1 and η^2 as

$$\phi = c_1, \qquad \xi = 0, \qquad \eta^1 = c_2\, v_1 + g(t,r) \text{ and } \qquad \eta^3 = c_2\, v_3 + h(t,r) \tag{21}$$

Here $g\,(t,r)$ and $h(t,r)$ are arbitrary functions of r of the following partial differential equations.

$$\mathrm{Re}\frac{\partial g}{\partial t} = \mu_2 \frac{\partial^2 g}{\partial r^2} + \frac{\mu_2}{r}\frac{\partial g}{\partial r} + \frac{\partial h}{\partial r} + \frac{h}{r} - \frac{1}{Da} g \quad \text{(a) and} \qquad We\frac{\partial h}{\partial t} = \mu_1 \frac{\partial g}{\partial r} - h \quad \text{(b)} \tag{22}$$

In (22), c_1 and c_2 are constants of integration. Thus the symmetry Lie algebra of the PDEs (13-i and 13-iii) is two–dimensional and defined by the following generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = v_1 \frac{\partial}{\partial v_1} + v_3 \frac{\partial}{\partial v_3} \;\; \& \qquad X_m = g(t,r)\,\frac{\partial}{\partial v_1} + h(t,r)\,\frac{\partial}{\partial v_3} \tag{23}$$

Where m is any natural number

3.3 Invariant Solution of the PDEs (13-i &iii) corresponding to Operator X_1 - α X_2

The form of invariant result related in the generator $X = X_1 - \beta X_2$ is given as

$$v_1(t,r) = e^{-\beta t}\,\phi_1(r) \quad \text{and} \quad v_3(t,r) = e^{-\beta t}\,\phi_3(r) \tag{24}$$

For bounded function, we must take exponential function in negative sign.
After putting the values of (25) into PDEs (13-i &13-iii) which gives the reduced ODEs system

$$\mu_2\,\phi_1''(r) + \frac{\mu_2}{r}\phi_1'(r) + \phi_3'(r) + \frac{1}{r}\phi_3(r) + (\beta\,\mathrm{Re} - \frac{1}{Da})\phi_1(r) = 0 \quad \text{(a)} \qquad \phi_3(r) = \frac{\mu_1}{1-\beta We}\phi_1'(r) \quad \text{(b)} \tag{25}$$

Put the value of $\phi_3(r)$ from (25-b) into (25-a), then we have,

$$\phi_1''(r) + \frac{1}{r}\phi_1'(r) + \lambda^2\,\phi_1(r) = 0 \qquad \text{Where } \lambda^2 = \frac{(\beta\,\mathrm{Re} - \frac{1}{Da})(1-\beta We)}{(1-\beta\,\mu_2\,We)} \tag{26}$$

Subject to boundary conditions $\phi_1(1) = 0$ and $\phi_1'(0) = 0$ (27)

The above ordinary differential equation (33) is the Bessel's differential equation whose order is zero and similarly equation have been solved as in section 3.1 and then the general solution of Bessel's differential equation is

$$\phi_1(r) = c_1\, J_0(\lambda r) + c_2\, Y_0(\lambda r) \tag{28}$$

Where $J_0(\lambda r)$ and $Y_0(\lambda r)$ are Bessel function of order zero of first and second kind respectively, i-e.

$$J_0(\lambda r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2\, 2^{2n}}\,(\lambda\, r)^{2n} \qquad \text{and } Y_0(r) = \frac{2}{\pi}[\, y_2(r) + (\gamma + \ln(2)\, j_0(\lambda r)$$

$$y_2(r) = \ln(r)\, j_0(\lambda r) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\varphi(n)}{2^{2n}\,(n!)^2}(\lambda r)^{2n} \tag{29}$$

Where γ is the co-efficient of combination and $\varphi(n) = [1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n}]$ for n=1, 2, 3, 4....... Hence $Y_0(\lambda r) \to -\infty$ when $r = 0$, so it is neglected, Therefore, the result is in one bounded solution is given as $\phi_1(r) = c_1\, J_0(\lambda r)$

After applying the boundary conditions from (28), so $\phi_1'(0) = 0$ is identically satisfied and

$\phi_1(r) = c_1\, J_0(\lambda) = 0 \Rightarrow c_1 \neq 0$, So $J_0(\lambda) = 0$, This has infinite number of roots λ_n $(n = 1, 2, 3, \ldots\ldots, \infty)$.

Hence applying the superposition principle, we obtain the solution $\phi_1(r)=\sum_{n=1}^{\infty} c_n\, J_0(\lambda_n\, r)$ (30-i)

and Substitute this value of $\phi_1(r)$ in equation (25-b), then $\phi_3(r)=\sum_{n=1}^{\infty}\frac{-\mu_1\,\lambda_n\, c_n}{(1-\beta We)}\,J_1(\lambda_n\, r)$ (30-ii)

As $\lambda^2=\frac{(\beta\,\mathrm{Re}-\frac{1}{Da})(1-\beta We)}{(1-\beta\,\mu_2\, We)}\Rightarrow\beta=\frac{1}{2}(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\ \mathrm{Re}})\pm\frac{1}{2}\sqrt{(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\ \mathrm{Re}})^2-\frac{4(Da\ \lambda_n^2+1)}{4(Da\ \lambda_n^2+1)}}$

As results are obtained after joint two equations and have same boundary positions, so for the time function, suppose

$$\beta_1=\frac{1}{2}(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})+\frac{1}{2}\sqrt{(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})^2-\frac{4(Da\,\lambda_n^2+1)}{Da\,\mathrm{Re}\,We}},$$

$$\beta_2=\frac{1}{2}(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})-\frac{1}{2}\sqrt{(\frac{1}{We}+\frac{\mu_2\,\lambda_n^2}{\mathrm{Re}}+\frac{1}{Da\,\mathrm{Re}})^2-\frac{4(Da\,\lambda_n^2+1)}{Da\,\mathrm{Re}\,We}} \tag{31}$$

Therefore, equation (25) expands into as under:

$$v_1(t,r)=\sum_{n=1}^{\infty}\left(c_{n_1}\,e^{-\beta_1 t}+c_{n_2}e^{-\beta_2 t}\right)J_0(\lambda_n\, r)\quad\text{and}\quad v_3(t,r)=\sum_{n=1}^{\infty}-\mu_1\,\lambda_n\left(\frac{c_{n_1}e^{-\beta_1 t}}{(1-\beta_1 We)}+\frac{c_{n_2}e^{-\beta_2 t}}{(1-\beta_2 We)}\right)J_1(\lambda_n\, r) \tag{32}$$

For the constants, applying the initial conditions (14-i) and (14-iii), so we obtain

$$c_{n_1}=\frac{2\,Da\ \ (1-\beta_1\,We)\,\beta_2}{(\beta_1-\beta_2)\,\lambda_n\,(Da\ \lambda_n^2+1)\,J_1(\lambda_n)}\qquad\text{and}\quad c_{n_2}=\frac{-2\,Da\ \ (1-\beta_2\,We)\,\beta_1}{(\beta_1-\beta_2)\,\lambda_n\,(Da\ \lambda_n^2+1)\,J_1(\lambda_n)} \tag{33}$$

3.3.1 Solution of PDE (13-ii)

As PDE is (13-ii), we have $We\,v_{2t}=2We\{(v_3+\psi_3(r))\,v_{1r}+\psi_1'(r)\,v_3\}-v_2$

As $\psi_3(r)=i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i}{\sqrt{Da}}r)}{J_0(\frac{i}{\sqrt{Da}})}\right)$, Consider $J_1(\frac{i}{\sqrt{Da}}r)=\sum_{n=1}^{\infty}c_n\,J_1(\lambda_n\,r)\Rightarrow c_n=\frac{2i\sqrt{Da}\ J_0(\frac{i}{\sqrt{Da}})}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}$ When $J_0(\lambda_n)=0$

So $J_1(\frac{i\,r}{\sqrt{Da}})=\sum_{n=1}^{\infty}\frac{2i\ \sqrt{Da}\ \ J_0(\frac{i}{\sqrt{Da}})}{(Da\,\lambda_n^2+1)\,J_1(\lambda_n)}\,J_1(\lambda_n\, r)$

After setting the value of $\psi_3(r)$ and according to the solutions (32), solution of PDE (13-ii) is agreed as

$$v_2(t,r)=\left(\sum_{n=1}^{\infty}\frac{Da\,(8\,We\ \ \mu_1)^{\frac{1}{2}}}{(Da\ \lambda_n^2+1)\,J_1(\lambda_n)}\left(\begin{array}{l}\frac{\beta_2^2(1-\beta_1 We)\ e^{-2\beta_1 t}}{(\beta_1-\beta_2)^2\ (1-2\,\beta_1 We)}+\frac{\beta_1^2(1-\beta_2 We)\ e^{-2\beta_2 t}}{(\beta_1-\beta_2)^2\,(1-2\,\beta_2 We)}+\frac{\beta_2\,(2-\beta_1 We)\ e^{-\beta_1 t}}{((\beta_1-\beta_2)(1-\beta_1 We)}\\-\frac{\beta_1\,(2-\beta_2 We)\ e^{-\beta_2 t}}{(\beta_1-\beta_2)(1-\beta_2 We)}-\frac{\beta_1\,\beta_2\,(2-\beta_1 We-\beta_2 We)\ e^{-(\beta_1-\beta_2)t}}{(\beta_1-\beta_2)^2(1-\beta_1 We-\beta_2 We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2+e^{\frac{-t}{We}}\,\varphi(r)$$

Now applying the initial condition (14-ii), then we obtain,

$$\varphi(r)=2We\,\mu_1\,Da\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)^2-\left(\sum_{n=1}^{\infty}\frac{Da\,(8\,We\ \ \mu_1)^{\frac{1}{2}}}{(Da\ \lambda_n^2+1)\,J_1(\lambda_n)}\left(\begin{array}{l}\frac{\beta_2^2(1-\beta_1 We)\ e^{-2\beta_1 t}}{(\beta_1-\beta_2)^2\ (1-2\,\beta_1 We)}+\frac{\beta_1^2(1-\beta_2 We)\ e^{-2\beta_2 t}}{(\beta_1-\beta_2)^2\,(1-2\,\beta_2 We)}+\frac{\beta_2\,(2-\beta_1 We)\ e^{-\beta_1 t}}{((\beta_1-\beta_2)(1-\beta_1 We)}\\-\frac{\beta_1\,(2-\beta_2 We)\ e^{-\beta_2 t}}{(\beta_1-\beta_2)(1-\beta_2 We)}-\frac{\beta_1\,\beta_2\,(2-\beta_1 We-\beta_2 We)\ e^{-(\beta_1-\beta_2)t}}{(\beta_1-\beta_2)^2(1-\beta_1 We-\beta_2 We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2 \tag{34}$$

Hence the final result of the system (4 to 6) admit the following solutions

$$u(t,r)=\sum_{n=1}^{\infty}\frac{2\,Da}{\lambda_n\,(Da\,\lambda_n^2-1)\,J_1(\lambda_n)}\left(\frac{(1-\beta_1 We)\,\beta_2\,e^{-\beta_1 t}}{(\beta_1-\beta_2)}-\frac{(1-\beta_2 We)\,\beta_1\,e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right)J_0(\lambda_n\, r)+Da\left(1-\frac{J_0(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) \tag{35-i}$$

$$\tau_{11}(t,r)=\left(\sum_{n=1}^{\infty}\frac{Da\,(8We\;\mu_1)^{\frac{1}{2}}}{(Da\;\lambda_n^2+1)\,J_1(\lambda_n)}\left(\begin{array}{l}\dfrac{\beta_2^{\,2}(1-\beta_1 We)\,(e^{-2\beta_1 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2\;(1-2\,\beta_1 We)}+\dfrac{\beta_1^{\,2}(1-\beta_2 We)\;(e^{-2\beta_2 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2\;(1-2\,\beta_2 We)}\\ -\;\dfrac{\beta_1\,\beta_2(2-\beta_1 We-\beta_2 We)\,(e^{-(\beta_1+\beta_2)t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2\;(1-\beta_1 We-\beta_2 We)}\\ +\dfrac{\beta_2\,(2-\beta_1\;We)\,(e^{-\beta_1 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)\;(1-\beta_1 We)}-\dfrac{\beta_1\;(2-\beta_2 We)\,(e^{-\beta_2 t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)\;(1-\beta_2 We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2 \tag{35-ii}$$

$$+\;2\,We\;\mu_1\,Da\left(e^{\frac{-t}{We}}-1\right)\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right)$$

$$\tau_{12}\,(t,r)=\sum_{n=1}^{\infty}\frac{-2\,\mu_1 Da}{(Da\,\lambda_n^{\,2}+1)\,J_1(\lambda_n)}\left(\frac{\beta_2\,e^{-\beta_1 t}}{(\beta_1-\beta_2)}-\frac{\beta_1\,e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right)J_1(\lambda_n r)+i\,\mu_1\sqrt{Da}\left(\frac{J_1(\frac{i\,r}{\sqrt{Da}})}{J_0(\frac{i}{\sqrt{Da}})}\right) \tag{35-iii}$$

As we have $J_0(\lambda_n)=0$, the graph of this equation is

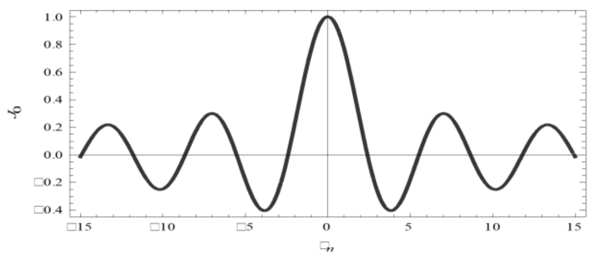

Fig.1: Graph of the relation $J_0(\lambda_n)=0$

Hence **(Fig. 1)** shows the different values of $\lambda_n=\pm 2.404825557,\ \pm 5.520078112,\pm 8.6537279,\pm 11.7915344,\ \ldots\ldots\ldots\ldots,\infty$

These values satisfy the equation $J_0(\lambda_n)=0$, hence we choose the one value $\lambda=2.404825557$ for the graph.

These values satisfy the equation $J_0(\lambda_n)=0$, so we select the one value $\alpha=\lambda=2.404825557$ for the graphs.

2. ANALYSIS OF VELOCITY

4.1 Graph of Invariant Solution of Velocity of Equation (36-i)

The analytical solutions of velocity have been obtained by Lie group method and are written in the equation (36-i) and plotted in **(Fig. 2)** for several parameters with $\mathrm{Re}=1,\ We=1,\ Da=10,\ \mu_1=\frac{1}{9}$, $\mu_2=\frac{8}{9}$, β_1 and β_2 and at different values of time *t*.

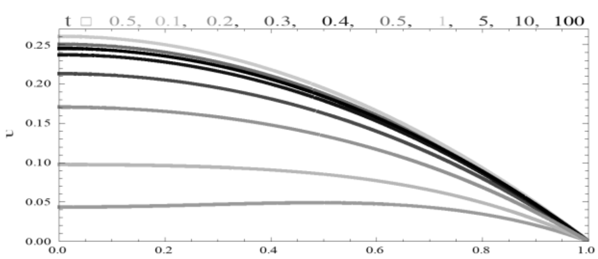

Fig.2: Analytical solution of the velocity *u* of (36-i) with *Re* = 1, $\mu_2=\frac{8}{9}$, *Da*=10 and at different values of time *t*.

The result of analytical solution of the velocity *u* of time dependent equation (36-i) is presented in figure–2 respectively. **(Fig. 2)** shows if time continues from rest, then pipe velocity profile enlarges and attained at maximum value of *u* = 0.261 and then some level decreases from the value of *u* = 0.261. At high level of time, flow become steady state at a value which is equal to 0.24555 and no further change in velocity profile.

4.2 Graphs of Study State Solution of Velocity.

The invariant solution related with X_1 is the steady-state solution of the velocity *u* which is already solved in the section (3.1). The result of steady state solution of system of ODEs (8) with the boundary conditions (9) is obtained in the relation (12) and steady-state solution of the velocity *u* is given as

$$u(t,r)=\psi_1(r)=Da\left(1-\frac{j_0(\frac{i\,r}{\sqrt{Da}})}{j_0(\frac{i}{\sqrt{Da}})}\right)=Da\left(1-\frac{\sum_{n=0}^{\infty}(4Da)^{-n}(n!)^{-2}r^{2n}}{\sum_{n=0}^{\infty}(4Da)^{-n}(n!)^{-2}}\right) \tag{37}$$

The graphs of steady state solutions of the velocity *u* are plotted in **(Fig. 3)** at different values of Darcy's number *Da*.

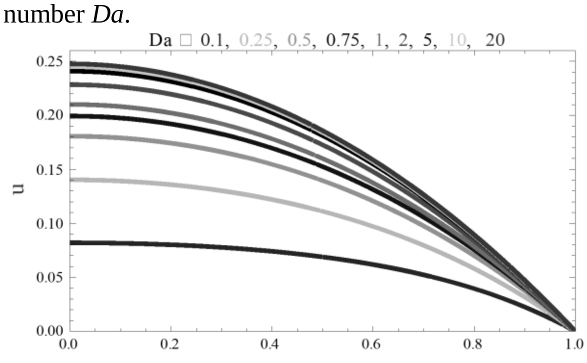

Fig.3: Steady state solution of the velocity *u* (37) at different values of *Da*.

The steady state velocity is displayed in fig.3 respectively. The **(Fig. 3)** shows that in the steady state, if pipe flows having small Darcy's numbers Da, then steady velocity *u* have small values that if permeability decreases i-e Darcy's number (porosity) of porous media decreases then resistance increases and hence velocity decreases in the steady state.

5 CONCLUSIONS

In this research paper, the activities of transient hydrodynamics relative with flow of the viscoelastic fluid in pipes filled with porous medium in conjunction of the constant viscosity Oldroyd–B constitutive model are researched. Hence the analysis is lectured interested to solve the problem of PDE's system for the analysis of velocity in analytical solution of velocity and Lie group technique has been applied successfully for solving PDEs of viscoelastic fluid flow and also numerical. The transformation group is a theoretic approach which is used to find the solutions of the problem. The one number of independent variables has been reduced through one-parameter group transformation and the PDE's system reduces to an ODE's system and the analytical solutions are obtained. The purpose of the current research is to obtain the exact analytical result of velocity adopting Lie group technique. We hope that the results may be useful for other workers in the field. Our advices for the future work are developing and putting into practice other steady-state and transient viscoelastic Algorithms.

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