SindhUniv. Res. Jour. (Sci. Ser.) Vol. 52 (02) 213 -220 (2020)

http://doi.org/10.26692/sujo/2020.06.17

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

Column Penalty Method for Solving assignment Problems

R. M. AHMED*, Z. U. A. KHUHRO**, F. N. MEMON***, A.S. SOOMRO****

Institute of Mathematics and Computer Science, University of Sindh, Jamshoro, Sindh, 76080, Pakistan

Received 10th January 2019 and Revised 15th May 2020

Abstract: The Assignment Problem is one of the combinatorial essential optimization problems in the field of Operations Research. The objective of the assignment problem is to reduce the time or cost of complement a number of activities by a number of resources. The main characteristic is equality between the number of sources and the number of destinations. In this paper, Column Penalty Method (CPM) for solving assignment problem is proposed. It can be seen that the results obtained by the proposed Column Penalty method reduces order of the matrix, number of tables and the number of iterations. It is an efficient method that will reduce steps in order to get optimal solution. The result of suggested method is also compared with the well known Hungarian Method and other existing methods. This method can also be used to solve any real physical world problem such as production planning, telecommunication design, traveling salesman problem, and the Airline Crew Assignment Problem (ACAP).

Keywords: Assignment Problem . Column Penalty Method . Efficient Method . Optimal Solution.

1 INTRODUCTION

Assignment problem is a section of operations or optimization research in mathematics. It is the particular conditions of transportation problems. The objective of the assignment problem is to reduce the time or cost of complement a number of resources by a number of activities. The significant characteristic of the assignment problem is the number of sources and the number of destinations should be equal Kuhn, H.W.[3]. It can be described as a square matrix where number of rows should be equal number of columns. If it is not equal then dummy rows or columns will be introduced in order to make it balanced problem. Assignment problem is appearing in various fields in our daily life. These problems have applications in production planning, telecommunication, economics, and employees to jobs, the airline fights, Crew Assignment Problems and in other different areas.

This study is to introduce a method called Column Penalty Method (CPM) to solve assignment problems. The existing methods take many iterations to get optimal solution such as Robust Ranking Techniques (RRT) [5], Genetic Assignment Method (GAM) [6], Ones Assignment Method (OAM) [1], an efficient Method for Assignment Problem [7], A Modified Approach for Assignment Method [2] and A Simple Method for Solving Fully Intuitionist Fuzzy Real Life Assignment Problem [4] therefore the column penalty method (CPM) has been suggested in order to reduce iterations, save time and to find the assignment problems optimum solution in an efficient way.

2 MATHEMATICAL MODEL OF ASSIGNMENT PROBLEM

Let C_{ij} be the value or cost of the assigning i^{th} resource to j^{th} task. The cost for n X n matrix is as follows:

$$
\begin{matrix}
C_{11}\ C_{12} \ldots C_{1n} \\
C_{21}\ C_{22} \ldots C_{2n} \\
\vdots \\
C_{n1}\ C_{n2} \ldots C_{nn}
\end{matrix}
$$

An assignment is a group of n entree locations in the cost matrix, no two of which lie in the same row or column. The cost of an assignment is the sum of the n entrees. An optimum assignment is an assignment with the minimal (maximal) possible cost. Here the assignment problem mathematical pattern is

*reem_mon2015@yahoo.com, **zain@usindh.edu.pk, ***farhatnaureen@gmail.com, ****dr_sattarsoomro@yahoo.co.in

described. Suppose that there are ***n*** missions and ***n*** machines. These ***n*** missions must be perfected by ***n*** machines, where the cost based on the specific work assignment. The following are some notations:
x_{ij} = The number of cells to be distributed from resource ***i*** to activity ***j***
Where (***i*** =1, 2, … , ***n*** and ***j*** =1, 2, …, ***n***);
C_{ij} = (non zero cost) cost or value per cell distributed from resource ***i*** to activity ***j.***

A generalized assignment model can be defined in the following way:
Let x_{ij} = 1, if i^{th} job is assigned to the j^{th} machine and 0, if i^{th} job is not assigned to the j^{th} machine.
Then the model is given by

Minimization

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij} \qquad \dots (1)$$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1,2,\dots,n \qquad \dots (2)$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad j = 1,2,\dots,n \qquad \dots (3)$$

$$x_{ij} \in \{0,1\} \quad \textit{for} \quad \boldsymbol{i, j} = 1,2,\dots$$

This paper proposes Column Penalty Method for finding the assignment problem optimum solution.
Also;
Minimization

$$\boldsymbol{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij}\ , \qquad \boldsymbol{i}, \boldsymbol{j} = 1,2,\dots,\boldsymbol{n}$$

where; $\boldsymbol{\alpha_{ij}}$ = minimum value in each row.
$\boldsymbol{\beta_{ij}}$ = maximum value in corresponding column.
$\boldsymbol{q_{ij}}$ = highest difference between $\boldsymbol{\beta_{ij}}$ and $\boldsymbol{\alpha_{ij}}$ in corresponding column,
Subject to

$$\boldsymbol{q_{ij}} = \sum_{i=1}^{n} \beta_{ij} - \sum_{j=1}^{n} \alpha_{ij} \qquad \dots (4)$$

Where $\beta_{ij} \geq \alpha_{ij} > 0, \quad x_{ij} = 1$

$$\boldsymbol{C_{ij}} = \begin{cases} \boldsymbol{q_{ij}}, & \boldsymbol{q_{ij}} > 0 \\ 0, & \textit{Otherwise} \end{cases}$$

2.1 PROPOSED ALGORITHM COLUMN PENALETY METHOD (CPM) FOR SOLVING ASSIGNMENT PROBLEM

A column penalty method for solving assignment problem has been developed. This method is based on [2].
Some modifications are being done in order to make it an efficient method.
Let X_1, X_2, X_3,...,X_n denote resources and Y_1, Y_2, Y_3,...,Y_n denote the activities. The following steps are proposed for finding solution of assignment problem.
Step 1: Structure the assignment problem cost matrix or table. Deem rows as a resources and columns as activities.

Y_1 Y_2 Y_3 . . . Y_n
X_1
X_2
X_3
⋮
X_n

Step 2: Form two columns, where resources appear in first column and activities appear in second column.

Column 1	Column 2
⋮	⋮
Xn	Yn
⋮	⋮

Step 3: Below first column, record the resources X_1, X_2, $X3$,..., X_n . Then find minima (maxima) unit cost for each resource, whichever minima (maxima) value or cost is available in the respecting column, choose it and record it in terms of activities below second column.
Continue this process for all the resources and record it in terms of activities.

Column 1	Column 2
X_1	Y_{min}
X_2	Y_{min}
⋮	⋮
X_n	

Y_{min} (in case of minimization),
Y_{max} (in case of maximization)

Step 4: For every resource if there is a single or unique activity (no other resource has minima (maxima) cost value in this activity) then assign that activity for the corresponding resource and delete this activity column and resource row, hence get the optimal solution.

Step 5: Insomuch as no single or unique activity for corresponding resource (there are two or more resources have minima (maxima) cost value in one activity) then: Check which rows have minima (maxima) unit cost in same activity. Next find difference among minima (maxima) and maxima (minima) unit cost at that activity column. Assign the activity which has maximal distinction. Delete this column and corresponding row.

However if there is equalize in distinction for two and more activities then, take the difference among minima (maxima) and next to maxima (next to minima) unit cost or if it is not available take the difference between minima (maxima) and zero. Therewith assign the activity which has maximum difference.

Step 6: Again find the minima (maxima) unit cost for the residual resources. Then iterate steps 4 to 5 until all resources are assigned uniquely to the corresponding activities (any resource assign to one and only one activity).

Step 7: Once all the resources are assigned then use the expression to calculate the total cost.
Total cost =

$$\boldsymbol{Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij}}$$

3 <u>WELL KNOWN CASES SOLVED BY CPM METHOD</u>

The proposed algorithm has been applied on various types of examples. It can be seen that the algorithm works well on all types of examples.

3.1 BALANCED ASSIGNMENT PROBLEM

3.1.1 Minimization Case:

Table 1 An example of balanced assignment is given where the minimization is required

	G	H	I	J	K	L
A	9	8	7	5	3	10
B	6	4	2	8	7	9
C	8	10	9	6	4	11
D	9	6	5	4	1	11
E	3	5	6	7	11	8
F	2	4	3	5	8	9

Solution of the problem given in Table 1using Column Penalty Method:

Form two columns, first column for resources and second column for activities that have minimum cost value.

Column 1	Column 2
A	K
B	I
C	K
D	K
E	G
F	G

B has unique activity at ***I*** because no any other resource has minimum cost value at activity ***I***. Assign **B** to ***I*** directly and delete that row and column.

	G	H	J	K	L
A	9	8	5	3	10
C	8	10	6	4	11
D	9	6	4	1	11
E	3	5	7	11	8
F	2	4	5	8	9

Column 1	Column 2	Difference
A	K	11 - 3 = 8
C	K	11 - 4 = 7
D	K	11 - 1 = 10

D has a maximum difference, so assign ***D*** to ***K*** and delete that row and column. Again find the minimum cost value for relative resources.

	G	H	J	L
A	9	8	5	10
C	8	10	6	11
E	3	5	7	8
F	2	4	5	9

Column 1	Column 2	Difference
E	G	9 - 3 = 6
F	G	9 - 2 = 7

F has a maximum difference, assign ***F*** to ***G*** and delete that row and column. Again find the minimum cost value for relative resources

	H	J	L
A	8	5	10
C	10	6	11
E	**5**	7	8

Column 1	Column 2	Difference
A	*J*	6 - 5 = 1
C	*J*	6 - 6 = 0
E	*H*	

E has unique activity at ***H*** assign it and delete that row and column; then find the difference minimum value of ***A*** *and* ***C*** and maximum value in corresponding column, ***A*** has a maximum difference, so assign ***A*** to ***J*** and ***C*** to ***L***.

Optimal Solution is

Resource	Activity	Cost
A	*J*	5
B	*I*	2
C	*L*	11
D	*K*	1
E	*H*	5
F	*G*	2
	Total	26

3.1.2 Maximization Case:

Table 2 An example of balanced assignment is given where the maximization is required

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
A	9	14	54	12	62	55	98	95
B	35	59	**82**	32	11	5	11	16
C	48	**87**	82	29	28	72	22	27
D	69	8	1	76	57	1	77	70
E	83	51	59	18	**84**	81	37	75
F	27	41	4	62	29	**89**	18	1
G	3	47	61	27	3	91	99	26
H	**86**	28	9	10	16	1	49	20

Solution of the problem given in Table 2 using Column Penalty Method: Form two columns, first column for resources and second column for activities that have maximum cost value.

Column 1	Column 2
A	7
B	3
C	2
D	7
E	5
F	6
G	7
H	1

B, C, E, F and ***H*** have unique activity at **3, 2, 5, 6** and **1**, assign them directly and delete those rows and columns.

	4	**7**	**8**
A	12	98	95
D	76	77	70
G	27	99	26

Column 1	Column 2	Difference
A	7	98 - 77 = 21
D	7	77 - 77 = 0
G	7	99 - 77 = 33

G has a maximum difference, assign ***G*** to **7** and delete that row and column.

	4	**8**
A	12	**95**
D	**76**	70

Again find the minimal cost value for remaining resources, ***A*** and ***D*** have unique minimal cost value at **8** and **4**

Obtained optimal solution is as follows:

Resource 1	Activity 2	Cost
A	8	95
B	3	82
C	2	87
D	4	76
E	5	84
F	6	89
G	7	99
H	1	86
	Total	698

3.2 UNBALANCED ASSIGNMENT PROBLEM

The method of assignment debated above demands the equality in rows and columns number in the assignment table or matrix. However, if the given cost matrix is not a balanced problem. In such cases a dummy column(s) or row(s) is (are) added in the problem matrix (with zeros cost elements) to make it *n X n* square matrix. Once the unbalanced assignment problem is converted into balanced assignment problem then the assignment problem can be solved.

Remarks: Dummy columns/rows will not be deemed as a minimal cost value.

3.2.1 Minimization Case:

An example of unbalanced problem is solved by Column Penalty Method. The problem is of minimization nature.

Table 3 An example of unbalanced assignment is given where the minimization is required

	A	B	C	D	E
1	5	7	11	6	5
2	8	5	5	6	5
3	6	7	10	7	3
4	10	4	8	2	4

Solution of the problem given in Table 3 using Column Penalty Method: The given cost matrix is unbalanced; hence add one dummy row with a zero cost.

	A	B	C	D	E
1	5	7	11	6	5
2	8	5	5	6	5
3	6	7	10	7	3
4	10	4	8	**2**	4
5	0	0	0	0	0

Form two columns, first column for resources and second column for activities that have minimum cost value for that resource.

Column 1	Column 2
1	*A*, *E*
2	*B*, *C*, *E*
3	*E*
4	*D*

4 have a unique activity at ***D***, assign **4** to ***D*** directly and delete that row and column.

	A	B	C	E
1	5	7	11	5
2	8	5	5	5
3	6	7	10	3
5	0	0	0	0

Column 1	Column 2	Difference
1	A, E	5 - 5 = 0
2	B, C, E	5 - 5 = 0
3	E	5 - 3 = 2

3 have a maximum difference, assign **3** to **E** and delete that row and column.

	A	B	C
1	**5**	7	11
2	8	5	5
5	0	0	0

Again find the minimum cost value for relative resources

Column 1	Column 2
1	*A*
2	*B*, *C*

Assign directly **1** to ***A*** and **2** to ***B***, here get optimal solution.

Obtained optimal solution is as follows:

Column 1	Column 2	Cost
1	*A*	5
2	*B*	5
3	*E*	3
4	*D*	2
	Total	15

3.2.2 Maximization Case:

Table 4 An example of unbalanced assignment is given where the minimization is required

	1	2	3	4	5
A	39	70	23	24	2
B	96	90	39	73	5
C	52	16	40	26	2
D	65	66	47	84	6
E	60	60	12	45	18
F	91	35	83	60	95
G	49	54	62	61	22

Solution of the problem given in Table 4 using Column Penalty Method: The given cost matrix is unbalanced; hence add two dummy columns with a zero cost to make it a square matrix.

	1	**2**	**3**	**4**	**5**	**6**	**7**
A	39	70	23	24	2	0	0
B	96	90	39	73	5	0	0
C	52	16	40	26	2	0	0
D	65	66	47	**84**	6	0	0
E	60	60	12	45	18	0	0
F	91	35	83	60	**95**	0	0
G	49	54	**62**	61	22	0	0

Write two columns, column 1 for resources and column 2 for activities that have maximum cost value.

Resource	Activity
A	2
B	1
C	1
D	4
E	1, 2
F	5
G	3

D, F and ***G*** have unique activity at **4, 5** and **3**, assign them directly and delete those rows and columns.

	1	**2**	**6**	**7**
A	39	70	0	0
B	96	90	0	0
C	52	16	0	0
E	60	60	0	0

Resource 1	Activity	Difference
A	2	70 - 16 = 54
E	1, 2	60 - 16 = 44

A has a maximum difference, assign ***A*** to **2** and delete that row and column.

	1	**6**	**7**
B	96	0	0
C	52	0	0
E	60	0	0

Again find the maximum cost value for relative resource.

Resource	Activity	Difference
B	1	96 - 52 = 44
C	1	52 - 52 = 0
E	1	60 - 52 = 8

B has a maximum difference, assign ***B*** to **1** and ***C, E*** to **6, 7**.

Optimal solution is

Resource	Activity	Cost
A	2	70
B	1	96
C	6	0
D	4	84
E	7	0
F	5	95
G	3	62
	Total	407

3.3 FUZZY ASSIGNMENT PROBLEM

3.3.1 Minimization Case:

Table 5 An example of fuzzy assignment is given where the minimization is require

(4,6,7,9)	(3,5,7,10)	(5,7,10,12)	(3,4,6,9)	(4,5,7,10)
(2,3,5,9)	(5,7,9,13)	(4,6,9,12)	(5,6,7,10)	(2,3,5,7)
(7,9,10,12)	(6,7,9,10)	(7,9,10,3)	(6,7,10,13)	(7,10,13,14)
(4,5,7,9)	(5,7,12,15)	(7,9,13,15)	(2,4,10,13)	(5,7,10,14)
(4,10,13,15)	(4,7,9,13)	(2,3,10,14)	(3,7,10,13)	(4,7,10,14)

Solution of the problem given in Table 5 using Column Penalty Method: Form two columns, first column for resources (rows) R_1 to R_5 and second column for activities (columns) C_1 to C_5 that have minimum cost value for that resource.

Column 1	Column 2
R_1	C_4
R_2	C_5
R_3	C_2
R_4	C_1
R_5	C_2

$\boldsymbol{R_1}$**,** $\boldsymbol{R_2}$ and $\boldsymbol{R_4}$ have a unique activity at $\boldsymbol{C_4}$**,** $\boldsymbol{C_5}$ and $\boldsymbol{C_1}$, assign $\boldsymbol{R_1}$ to $\boldsymbol{C_4}$**,** $\boldsymbol{R_2}$ to $\boldsymbol{C_5}$ and $\boldsymbol{R_4}$ to $\boldsymbol{C_1}$ directly and delete those rows and columns.

(6,7, 9,10)	(7,9,10,13)
(4,7, 9,13)	(2,3,10,14)

Column 1	Column 2	Difference
R_3	C_2	(6,7,9,10) - (4,7,9,13) = (2,0,0,3)
R_5	C_2	(4,7,9,13) - (4,7,9,13) = (0,0,0,0)

$\boldsymbol{R_3}$ has a maximum difference, assign $\boldsymbol{R_3}$ to $\boldsymbol{C_2}$ and $\boldsymbol{R_5}$ to $\boldsymbol{C_3}$.

Obtained optimal solution is as follows:

Resource	Activity 2	Cost
R_1	C_4	(3, 4, 6, 9)
R_2	C_5	(2, 3, 5, 7)
R_3	C_2	(6, 7, 9, 10)
R_4	C_1	(4, 5, 7, 9)
R_5	C_3	(2, 3, 10, 14)
	Total	(17, 22, 37, 49)

4 RESULTS AND DISCUSSION

In this section balanced as well as unbalanced assignment problems are being solved by Hangarian method (HM) [3], Genetic Assignment Method (GAM)[6], Ones Assignment Method (OAM)[1], Robust Ranking Techniques (RRT)[5] and proposed (CPM) in order to compare the results and see the performance in the form of tables and iterations.

4.1 Balanced Assignment Problem

4.1.1 Minimization Cases:

This section shows examples of minimization balanced assignment problem in different orders. The Table 6 shows result of minimization balanced assignment problem using different existing methods as well as CPM.

Table 6 Some examples of balanced assignment problems of minimization are given and the results are compared with CPM and other well known methods

EX No.	Order	Tables				Result
		HM	CPM	OAM	GAM	
1	6x6	6	3	6	5	**26**
2	4x4	6	1	6	3	**15**
3	5x5	6	3	6	4	**24**
4	5x5	5	2	5	4	**23**
5	5x5	6	3	6	4	**26**

4.1.2 Maximization Cases:

This section presents some examples of maximization balanced assignment problem in different orders. The Table 7 shows results of maximization balanced assignment problem using different existing methods as well as CPM.

Table 7 Some examples of balanced assignment problems of maximization are given and the results are compared with CPM and other well known methods.

EX No.	Order	Tables				Result
		HM	CPM	OAM	GAM	
1	5x5	11	2	10	4	**419**
2	6x6	10	1	10	5	**305**
3	8x8	6	2	6	5	**698**
4	4x4	9	3	8	3	**26**
5	4x4	8	2	7	3	**27**

4.2 Unbalanced Assignment Problems

4.2.1 Minimization Cases:

This section shows some examples of minimization unbalanced assignment problem. The Table 8 shows results of minimization unbalanced assignment problem using different existing methods as well as CPM.

Table 8 Some examples of unbalanced assignment problems of minimization are given and the results are compared with CPM and other well known methods

Ex No.	Order	Tables				Results
		HM	CPM	OAM	GAM	
1	4x5	5	3	5	4	**15**
2	8x5	4	3	4	4	**870**
3	4x5	7	4	7	4	**10**
4	4x3	6	2	6	3	**130**
5	3x4	6	2	6	3	**54**

4.2.2 Maximization Cases:

This section presents some examples of maximization unbalanced assignment problem in different orders. The Table 9 shows results of maximization unbalanced assignment problem using different existing methods as well as CPM.

Table 9 Some examples of unbalanced assignment problems of maximization are given and the results are compared with CPM and other well known methods

EX No.	Order	Tables			Results
		HM	CPM	OAM	
1	4x7	7	2	7	**349**
2	7x6	12	4	11	**492**
3	4x5	10	4	10	**50**
4	5x4	6	5	6	**68**
5	4x6	9	4	9	**284**

4.3 Fuzzy Assignment Problem

4.3.1 Minimization Cases:

Some examples of fuzzy assignment problems of different orders are presented in this section. The Table 10 shows results of fuzzy assignment problem using different existing methods as well as CPM.

Table 10 Some examples of fuzzy assignment problem of minimization are given and the results are compared with CPM and other well known methods

EX No.	Order	Tables			Results
		HM	CPM	RRT	
1	5x5	6	2	6	(17,22,37,49)
2	4x4	5	2	5	(20,40,60)
3	4x4	5	2	5	(13,20,27)
4	3x3	5	1	5	(13,16,19)
5	4x4	6	2	6	(16,23,27,35)

5 SUMMAEY OF RESULTES

Performance of CPM and other existing methods is summarized in the owing Table 11.

Table 11 Comparison between CPM and other well known methods

Other Existing Methods	Column Penalty Method (CPM)
Complicated. It takes much iteration to get optimal solution. It depends on working in every element in the columns and rows of matrix. Order of the matrix is same in all steps of the solution. It takes so many tables during solution. You get optimal assignment of all resources after going throw all iterations. Original matrix changes during the solution. It takes long time to performing the tasks.	Easy to calculate and understand. It takes few steps to get optimal solution. It depends on working on one column of the matrix. It reduces the order of the matrix, so table of matrix get smaller step by step. It reduces number of tables during solution. You can get the optimal assignment of any resource directly from original matrix. It works on original matrix until get obtained solution. It takes short time to performing the tasks.

6 CONCLUTION

In the past, many researchers as well as practitioners used Hungarian method in order to solve assignment problems. The objective is to find an optimum assignment of persons to jobs without assigning a person more than time and ensuring that all missions are completed. Proposed Column Penalty Method presented with numerical examples to show how it works more efficiency than other methods. This CPM can be applied for effective optimum allocation of the combination problems which are related to combinatorial optimization and limited resources; hence it is possible to solve real-world major scale problems.

When comparing the result with Hungarian method and other methods it is found that existent methods take much iteration to find optimum solution therefore the column penalty method has been proposed in order to reduces iterations, save time and get optimum solution of assignment problem by easy way. It is a much faster and more efficient tool to handle the assignment problem than the Hungarian Method and other existing methods.

REFERENCES

1. Basirzadeh H (2012) Ones assignment method for solving assignment problems, Applied Mathematical Sciences, 6(47): 2345-2355
2. Khandelwal A (2014) A modified approach for assignment method, International journal of latest research in science and technology, 3(2):13-19
3. Kuhn HW (1991) On the origin of the Hungarian method, History of mathematical programming; a collection of personal reminiscences (J.K. Lenstra, A.H.G. Rinnooy Kan, and A. Schrijver, eds.), North Holland, Amsterdam, pp:77- 81
4. Kumar PS, Hussain, RJ (2016) A simple method for solving fully intuitionist fuzzy real life assignment problem, International Journal of Operations Research and Information Systems (IJORIS), 7(2):39-61
5. Nagarajan R, Solairaju A (2010) Computing improved fuzzy optimal Hungarian assignment problems with fuzzy
Costs under Robust Ranking Techniques, International Journal of Computer Applications, 6(4):6-13
6. Nizami ST, Khan JA, Khan FH, Khan N, Syed I (2011) A new method for finding the cost of assignment problem using genetic algorithm of artificial intelligence, International Journal of Latest Trends in Computing, 2(1): 135-137
7. Porchelvi SR, Anitha M (2018) Optimal solution for assignment problem by average total opportunity cost method, Journal of Mathematics and Informatics, 13:21-27