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Constraint Cycles in Quadratic Framework with Their Properties

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Abstract: In this study, an analysis is performed to study the reality and individuality of the limit cycle for a quadratic system. An algorithm is narrated for finding focal basis. These evaluations are then utilized to compute the small-amplitude limit cycle. The same approach is applied to the quadratic system and examples are formulated constructed for the small-amplitude limit cycle. For this quadratic system in the plane of two autonomous differential equations is considered, examples are also constructed by utilizing the technique of bifurcation of limit cycle & Poincare Bendixon theorem. Besides the number and distribution of limit cycle in the system are also discussed.

Keywords: Liapunov, Quantities; Constraint Cycles; Quadratic Framework; Effects

1. INTRODUCTION

Hilbert's sixteenth issue (Hilbert,1908) includes twofold parts. The principal portion, which ponders the shared mien of maximal range of partitioned branches of a logarithmic bend, conjointly the "corresponding investigation" for non-singular real logarithmic assortments: the moment portion, which possess the address of maximal number and relative position of constrain cycles of the polynomial system $\dot{\eta} = \Psi(\eta,\xi),\ \dot{\xi} = \Phi(\eta,\xi)$. Customarily, the primary portion of Hilbert's sixteenth issue is the subject of think about for pros of the genuine arithmetical geometry whereas the moment portion is explored by the mathematicians of standard differential conditions and dynamical framework. Li. Jibin, (2003) demonstrates that there exists conceivable association between these two parts; he speaks to the advance of considering on Hilbert's sixteenth issue and the relationship between Hilbert's sixteenth issue and bifurcation of planar vector fields.

In this research we contemplate the second portion of Hilbert's sixteenth concern. This portion consider the following questions:

1) Is it genuine that a planar polynomial vector field includes a limited number of constraining cycles?
2) Is it genuine that the number of restraining cycles of a planar polynomial vector field is bounded by a consistent depending on the degree of the polynomials only?
3) Is it genuine that the number of constraining cycles of a planar polynomial vector field is limited by a stable dependency on the degree of the polynomials only?
4) Given an upper bound on the number of restrainingcycles in 2.

Most of the mathematician works on the above issue. A few of them ponders the restrain cycles which bifurcate from occasional circles or middle for quadratic framework (Bautin 1952, Berlinsk 1960, Bomon 1987, Petrov 1988, Mardesic 1990, Yashenko 1991, Zoladek 1994, Horozov, *et. al.* 1994, Zhang, *et. al.* 1995, Guo, *et. al.* 1997, Gavrilov 2001). Be that as it may, it remains unsolved. In this paper, we concern with the quadratic framework of two independent differential systems within the plane and build illustration within the same framework by utilizing the strategy to bifurcate restrain cycle from the fine center and utilize of Poincare Bendixon hypothesis (Blows, *et. al.* 1984). We also talk about the number and dissemination of constraining cycle in the framework. For that following differential system is considered:

$$\left.\begin{aligned}\dot{\eta} &= \Psi(\eta,\xi)\\ \dot{\xi} &= \Phi(\eta,\xi)\end{aligned}\right\} \qquad (1)$$

where

$$\Psi(\eta,\xi) = \sum_{i+k=0}^{2} a_{ik}\eta^i\xi^k,\ \Phi(\eta,\xi) = \sum_{i+k=0}^{2} b_{ik}\eta^i\xi^k \qquad (2)$$

and a_{ik} , $b_{ik} \in R$.

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This paper is organized as follows. In the next section, we describe the procedure for determining the Liapunov function. Then we use the technique described in the section-2 to investigate the example constructed in the quadratic system and some properties of the quadratic framework will also be discussed.

2. ALGORITHM FOR CONSTRUCTION OF LIAPUNOV QUANTITIES

The estimated equations are utilized to approximate the central values μ_{2k} and Liapunovquantities. The approximated framework specified by (1) is described as follows:

$$\left.\begin{aligned}\dot{\eta} &= \lambda\eta + \xi + \sum_{i=1}^{i=n}\psi_i(\eta,\xi)\\ \dot{\xi} &= -x + \lambda\xi + \sum_{i=1}^{i=n}\phi_i(\eta,\xi)\end{aligned}\right\} \quad (3)$$

here, for*i* = *2,...,n,* ψ_i and ϕ_i are homogeneous polynomials of degree*i*.

We pursueto established the relation υ such that the rate of variation alongside the trajectory is $\mu_2\gamma^2 + \mu_4\gamma^4 + \cdots + \mu_{2k}\gamma^{2k} + \cdots$, where, $\gamma^2 = \eta^2 + \xi^2$ and $\mu_{2,}\mu_4, \ldots, \mu_{2k}$, are constants.

We transcribe

$$\nu(\eta,\xi) = \frac{1}{2}(\eta^2+\xi^2) + \sum_{k=3} \nu_k(\eta,\xi) + \ldots\ldots \quad (4)$$

where υ_k , for*k*≥3 is specified by

$$\nu_k(\eta,\xi) = \sum_{i=0}^{k}\nu_{k-i,i}\eta^{k-i}\xi^{i} \quad (5)$$

and $\nu_k$for the support vector $(\upsilon_{k,0},\upsilon_{k-1,1},\ldots,\upsilon_{0,k})^T$. The variation rate of υin a process is

$$\dot{\nu} = \left(\eta + \sum_{i=3}^{\infty}(\nu_i)_\eta\right)\left(\lambda\eta + \xi + \sum_{j=2}^{n}\psi_j\right) + \left(\xi + \sum_{i=3}^{\infty}(\nu_i)_\xi\right)\left(-\eta + \lambda\xi + \sum_{j=2}^{n}\phi_j\right) \quad (6)$$

here subscript η and ξ denote partial differentiation w. r. to of η and ξ, respectively.

The collected terms of the degree k in equations (6) are represented by $\omega_{k,}$ which for $k \geq 3$ are given in equation (7)

$$\begin{aligned}\omega_k &= [\xi(\upsilon_k)_\eta - \eta(\upsilon_k)_y]\\ &+ [(\upsilon_{k-1})_\eta\psi_2 + (\upsilon_{k-1})_\xi\phi_2\\ &+ \ldots\ldots + \eta\psi_{k-1} + \xi\phi_{k-1}]\end{aligned} \quad (7)$$

The coefficients of second part of ω_k can be can be expressed linearly in the terms of the coefficients in ψ_k and ϕ_k .

The indication to select coefficients of $\upsilon_{ij}$and quantities μ_k , such that

$$\omega_k = \begin{cases} 0 & \textit{for k is odd}\\ \mu_k(\xi^2+\eta^2)^{\frac{1}{2}k} & \textit{for k is even}\end{cases} \quad (8)$$

For suitability, we considered $\upsilon_{i,k-j}$asa variant coefficient of $\upsilon_k$analogues to *i*,to be even or else.

Assuming *k* to be odd, condition $\omega_k = 0$,is corresponding to *(k+1)th* group through *k+1* rectilinear qualities with $\upsilon_{k,0},\upsilon_{k-1,1},\ldots,\upsilon_{0,k}$ unknowns. These perturb into two groups of $\left(\frac{k+1}{2}\right)$equations, one of them figure out the odd-constraints of υ_k while the other provide even and accordingly υ_k is exclusively proven.

Considering evenpart (*that isfor k to be 2n*)the situation, $\omega_k = 0$ is identical to (2n+1)-rectiline are qualities which also provide two assortmentscomprises of (n+1)-collection having n-odd component of υ_k and *n*-equations involving the *(n+1)*-even groups. Subsequently $\omega_k = 0$, cannot be equally distributed, in order to overcome this concern another variable μ_k is introduced for which $\omega_k = \mu_k(\xi^2+\eta^2)^{k/2}$. It provides n-odd parameters of υ_k involving μ_k with n-equations. Onimposing supplementary condition such as:

(i) when k is multiple of 4, we set $\upsilon_{n,n} = 0$

(ii) when *k* ≡2 modulo4, we set

$$\upsilon_{n+1,n-1} + \upsilon_{n-1,n+1} = 0$$

Briefly for any type of given framework, following four stages are to be monitored to work out solution:

(a) Approximation of dominant quantities.
(b) Liapunov quantities are achieved by reduction of indispensableapproximation.
(c) Quantity,k_{max}, is assessed by determining origin at $\mu_{2k} = 0$, $\forall k \le k_{\max} + 1$
(d) Beginning with an adequate epicenter of utmost order, compute categorization of disorder such that each reverses steadiness of the origin.

In this study Liapunov quantities were estimated by utilization of Derive-6 package.The author commands based program is used whose working flow chart is depicted in**(Fig. 1)**.

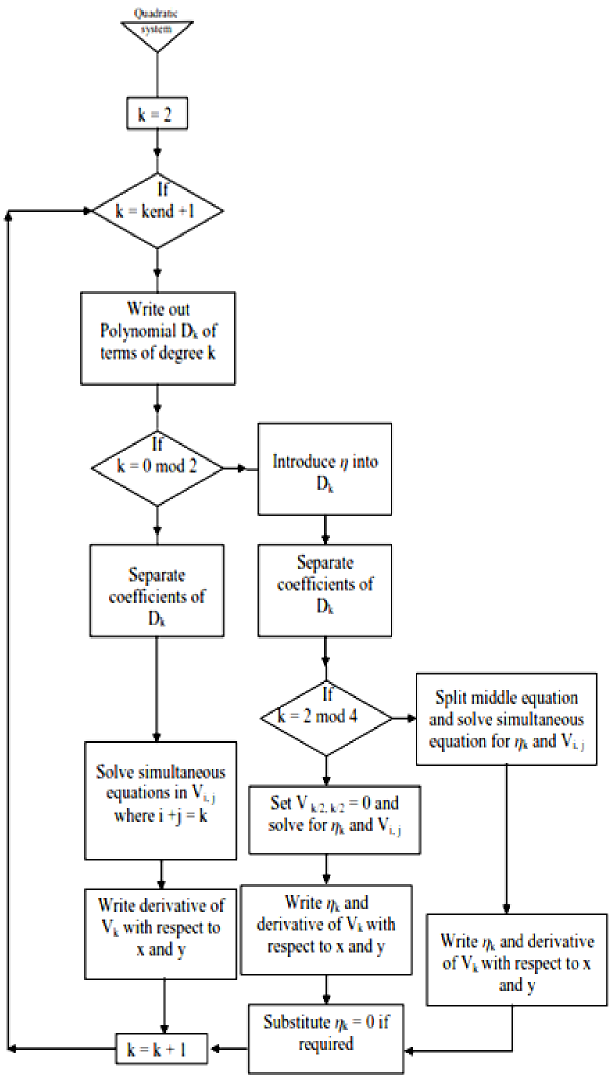

Fig.1. Liapunov quantities computation algorithms.

3. SOME SPECIAL QUADRATIC SYSTEMS AND THEIR PROPERTIES

Considering the quadratic system of 2-dimensional self-governing differential equations in the plane as specified by (1), we produce the subsequent two illustrations with maximumunique limit cycle motivated by (Berlinskii 1960, Gavrilov 2001).

3.1 Example 1*:*

$$\begin{aligned} \dot{\xi} &= \lambda\xi + \gamma - \alpha x^2 - 2\beta xy \\ \dot{\gamma} &= -\xi + \lambda\gamma + \beta y^2 \end{aligned} \qquad (9)$$

3.2 Solution:By following the procedure discussed above, estimated results as follows:

$$\mu_2 = \lambda \qquad (10)$$

and

$$\mu_4 = -\frac{1}{4}\xi\gamma \qquad (11)$$

3.3 Theorem 1:The Liapunovmeasures for the frame work illustrated by (8) are obtained from μ_{2k+2}, for $k \in \{0,1\}$ and are specified by

$$\left.\begin{aligned} K(0) &= \lambda \\ K(1) &= -\xi\gamma \end{aligned}\right\} \qquad (12)$$

These acquired quantities require two tasks: one of the requirement is imperative pivotal bases with group of calculation of small-amplitude with restricted possible cycle. We want to reflect only frame work for which K(0) = 0, we consider $\lambda = 0$.

To work concerning to the prerequisite conditions for the source to be a adequatecenter of order one, we suppose *K(1) = 0.*

Such that $\xi\gamma = 0$, with either $\xi = 0$ *or* $\gamma = 0$

3.4 Case 1: If $\xi = 0$, $\gamma \neq 0$

The frame wok (8) converts

$$\left.\begin{aligned} \dot{\eta} &= \xi - 2\gamma\,\eta\,\xi \\ \dot{\xi} &= -\eta + \gamma\,\xi^2 \end{aligned}\right\} \qquad (13)$$

then

$$\frac{\partial \Psi}{\partial \eta} + \frac{\partial \Phi}{\partial \xi} = 0 \qquad (14)$$

in which

$$\left.\begin{aligned} \Psi &= \xi - 2\gamma\,\eta\xi \\ \Phi &= -\eta + \gamma\,\xi^2 \end{aligned}\right\} \qquad (15)$$

Consequently, the source is center by divergence theorem.

3.5 Case 2:If $\gamma = 0$, $\alpha \neq 0$

Frame work described by (8) implies

$$\left.\begin{aligned} \dot{\eta} &= \xi - 2\alpha\,\eta^2 \\ \dot{\xi} &= -\eta \end{aligned}\right\} \qquad (16)$$

Subsequently the system (15) is symmetric about ξ-axis, therefore origin is center by symmetry principle. So we have the following lemma.

3.6 Lemma I:The source of scheme (8) is a wellcentered of order one if $\alpha\gamma \neq 0$ and it is at center if $\alpha\gamma = 0$.

3.7 Theorem 2:Suppose basis of structure (8) is a satisfactorycenter of order one. Then the inappropriate perturbation single small-amplitude constraint cycle can be createdat the basis.

3.8 Proof:Since the basis is a fine focus of unit order consequently, utilizing Lemma-I

$$\left.\begin{aligned} K(0) &= \lambda = 0 \\ K(1) &\neq 0 \end{aligned}\right\} \qquad (17)$$

For certainty, we assume that $\alpha, \beta > 0$. We discomposeβ so that $K(0) = 0$ but $K(0)K(1) < 0$, this is accomplished by increasing β. Thus the permanency of basis is inverted and a limit cycle bifurcate out of the adequatecenter which proved the statement of Theorem-I.

3.9 Example 2:Consider a newquadratic framework the form

$$\left.\begin{aligned} \dot{\eta} &= \lambda\eta + \xi + l\eta^2 + 10\alpha\eta\xi \\ \dot{\xi} &= -\eta + \lambda\xi + \alpha\eta^2 - 5\alpha\xi^2 \end{aligned}\right\} \qquad (18)$$

3.10 Solution:Using the algorithm, we have

$$\left.\begin{aligned} \mu_2 &= \lambda \\ \mu_4 &= -\alpha l \end{aligned}\right\} \qquad (19)$$

3.11 Theorem 3: The Liapunovmeasures of framework which are consequential from μ_{2k+2} , $k = 0,1$ are given by

$$\left.\begin{aligned} K(0) &= \lambda \\ K(1) &= -\alpha l \end{aligned}\right\} \qquad (20)$$

To calculate these values, we contain crucial pivotal centers and we want to display a type of equalities with as many small-amplitude constraint cycle as conceivable. We consequently, want to reveal the only framework for which $K(0) = 0$, for that we, assume $\lambda = 0$.

We compute stipulation for the starting point to be aadequate focus of order one, we assume,K(1) = 0, such that $\alpha l = 0$, either $\alpha = 0$ or $l = 0$

3.12 Case 1: If $\alpha = 0$, $l \neq 0$

Frame work (11) ⇒

$$\left.\begin{aligned} \dot{\eta} &= \xi + l\eta^2 \\ \dot{\xi} &= -\eta \end{aligned}\right\} \qquad (21)$$

This framework is consistent about η-axis, therefore origin is center by consistent principle.

3.13 Case 2: If $l = 0$, $\alpha \neq 0$

Frame wok (18) implies:

$$\left.\begin{aligned} \dot{\eta} &= \xi + 10\alpha\eta\xi \\ \dot{\xi} &= -\eta + \alpha\eta^2 - 5\alpha\xi^2 \\ \frac{\partial\Psi}{\partial\eta} &+ \frac{\partial\Phi}{\partial\xi} = 0 \end{aligned}\right\} \qquad (22)$$

where

$$\left.\begin{aligned} \Psi &= \xi + 10\alpha\eta\xi \\ \Phi &= -\eta + \alpha(\eta^2 - 5\xi^2) \end{aligned}\right\} \qquad (23)$$

Thus origin is center by Divergence theorem. So we have the following lemma.

3.14 Lemma 2:The source of frame work (17) is fine center of single order if $\alpha\, l \neq 0$ and it is a center if $\alpha\, l = 0$.

3.15 Theorem 4:Suppose origin of frame work (18) is a fine focus of single order. Then the under suitable perturbation one small amplitude limit cycle can be generated out of the origin.

3.16 Proof:Since the origin is a fine focus of single order consequently, by Lemma 2.

$$\left.\begin{aligned} K(0) &= \lambda = 0 \\ K(1) &\neq 0 \end{aligned}\right\} \qquad (24)$$

For assurance, we assume*al*> 0. We discompose'*a'* so that *K(0) = 0*however, *K(0)K(1) < 0*, this is attained by increasing α. Thus the steadiness of origin is reversed and a limit cycle bifurcates out of the sufficient focus source.

4. PROPERTIES OF QUADRATIC FRAMEWORK

Following are the generic properties of the quadratic framework

- In a quadratic framework a periodic orbit is curved and comprises a unique critical point in its interior.
- Two periodic orbits are oppositely oriented if external to each other and similarly oriented if one is inside the other
- In a quadratic frame work there are at most two shells of intermittent orbits.

6. CONCLUSION

In this study 2-dimensional impartial framework of the differential equations of the form $\dot{\eta} = \frac{d\eta}{dt} = \Psi(\eta, \xi), \ \dot{\xi} = \frac{d\xi}{dt} = \Phi(\eta, \xi)$ with Ψ and Φ as polynomials of real variables η and ξ are considered. From this developed framework examples with at most one restriction cycle has been developed. Besides, distinct common properties of the quadratic framework have also been reflected.

REFERENCES:

Bautin, N. N. (1952). On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. Vol. 30, 181-196.

Berlinskii, A. N. (1960). On the co-existing of singular point different types Izv. Vyss. UcebnZaved, Matematika. vol. 5, no. 18, 27-32.

Blows T. R and N. G. Lloyd, (1984). The number of limit cycles of certain polynomial differential equation, Proc. Royl. Soc. Edinburgh Sect. A 98, 215-239.

Bomon, R. (1987), Qualitative vector fields in the plane have a finite number of limit cycles, Publ. I. H. E. S. Vol. 64, 111-142.

Guo, C. and S. Quan, (1997). Limit cycles of a class of higher-degree polynomial systems and their applications, J. Lanzhou Univ. Nat. Sci. vol.3, no. 33,9-13.

Gavrilov, L. (2001). The infinitesimal 16th Hilbert problem in the quadratic case, Invent. Math., vol. 143, 449-497.

Hilbert, D. (1908). Mathematical Problem, Bulitten American Mathematical Scoiety, vol. 8, 437–779.

Horozov E. and I.D. Iliev, (1994). On the number of limit cycles in perturbation of quadratic Hamiltonian systems, Int..Proc of Math. vol. 69,London, 198-224.

Li. J., (2003). Hilbert.s problem and bifurcation of planar polynomial fields, Int. J. bifurcation and chaos., vol. 13, no. 1, 47–107.

MardesicP. (1990). The number of limit cycles of polynomials deformations of a Hamiltonian vector field, Ergodic theory & dynamical systems, vol.10, 523-529.

Petrov, G. S. (1988). Number of zeros of complete elliptic integrals, Funct. Anal. Appl, vol.18, 148-149.

Yu. S. I. and I. Yashenko, (1991). Finitiness theorem for limit cycles, Translation of Mathematical Monograph, Vol-94. Amer. Math. Soc.

Zoladek, H. (1994). Qualitative systems with centre and their perturbations, J. Differential Equations, vol. 109, 223-273.

Zhang, Z. and B. Li, (1995). "A note on a result on G.S. Petrov about the weakend Hilbert's 16th problem", J.Math. Anal. Appl., vol. 190, 489-516.