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Proper Curvature Collineations in Som-Roy Chaudhary Symmetric Space time

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Abstract: Rank of the 6×6 Riemann matrix and direct integration technique are used to investigate the proper curvature collineations(CCS) in the Som-Roy Chaudhary symmetric space-time. Studying proper (CCS) of the given spacetime, it is shown that there is only one case when the spacetime admits proper (CCS).These proper (CCS) form an infinite dimensional vector space.

Keywords: Rank of 6×6 Riemann matrix; Curvature collineations; Direct integration techinque, Infinite dimensional vector space

1. INTRODUCTION

The purpose of this paper is to investigate the proper (CCS) in Som-Roy Chaudhary symmetric space-time. The importance of the curvature symmetry cannot be ignored in Einstein's theory of general relativity and gravitation. Under this symmetry the curvature structure is also preserved. As well as the set of Einstein's Field Equations (EFEs) is concerned the geometric part of the (EFEs) is based on the curvature of the space-time. So to find the static and non-static solutions of (EFEs) it is, therefore, important to study curvature collineations (Stephani, *et al.*, 2003). Here an approach, which is introduced by (Hall and da Costa, 1991), is employed to study proper (CCS) in the above space-time. In this paper M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). Component form of Riemann curvature tensor and Ricci tensor which can be determined with given metric g_{ab}, by finding the Christoffel symbols are denoted by $R^a{}_{bcd}$ and $R_{ab}=R^c{}_{acb}$ respectively. Here comma and semicolon represent the partial and covariant derivatives respectively. The symbol L is used for Lie derivative. If the curvature tensor does not vanish over any non-empty open subset of manifold M, then manifold is considered to be non-flat. Furthermore, covariant derivative of any vector field X on the manifold can be splitted into symmetric and skew symmetric parts as

$$X_{a;b}=\frac{1}{2}h_{ab}+F_{ab}, \qquad (1.1)$$

where $h_{ab}=h_{ba}=L_X g_{ab}$ is a symmetric and $F_{ab}=-F_{ba}$ is a skew symmetric tensor on M. If $h_{ab;c}=0$ then vector field is said to be *affine.* On the other hand if $L_X g_{ab}=2\alpha g_{ab}, \alpha\in R$ then X is called *homothetic.* The metric preserving transformation is called isometry or *Killing vector field* i.e $L_X g_{ab}=0$. The vector field which is not homothetic is said to be proper affine. if it is not homothetic vector field . On the other hand if it is not Killing vector field then X is said to be proper homothetic.

If the Riemann curvature tensor is conserved along the vector field X on M then X is called (CCS) (Katzin, *et al.*,1969, Hall, 2004) i.e

$$L_X R^a{}_{bcd}=0. \qquad (1.2)$$

The expansion of above equation (1.2) is

$$R^a{}_{bcd;e}X^e+R^a{}_{ecd}X^e{}_{;b}+R^a{}_{bed}X^e{}_{;c}$$
$$+R^a{}_{bce}X^e{}_{;d}-R^e{}_{bcd}X^a{}_{;e}=0.$$

For a vector field X to be proper (CC), it should not be affine (Hall, 2004) on M. The compact representation of Eq.(1.2) into set of twenty two coupled nonlinear partial differential equations is given by (Bokhari, *et al.*, 2003).

2. CLASSIFICATION OF THE RIEMANN TENSORS

This section is devoted to classify the Riemann tensor by its rank and bivector decomposition. The rank of the 6×6 symmetric Riemann matrix derived in an eligent way in (Hall and da Costa, 1991). The rank of

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the Riemann tensor at the point p on the maniofld can also be determined by the rank of the linear map $\tau : F^{ab} \rightarrow R^{ab}{}_{cd} F^{cd}$. Define the subspace S_p of the tangent space $T_p M$ which consists of those members u of $T_p M$ which satisfy the Ricci identity

$$R_{abcd}\, u^d = 0. \qquad (2.1)$$

Then following algebraic conditions by the Riemann tensor at point p in terms of bivector decomposition are satisfied (Hall, 2004).

Class B

If the range of τ is spanned by the dual pair of non-null simple bivectors and $\dim S_p = 0$. Then rank of Riemann tensor is 2 and at p it takes the form

$$R_{abcd} = \alpha\, F_{ab} F_{cd} + \beta \overset{*}{F}_{ab} \overset{*}{F}_{cd}, \qquad (2.2)$$

where F and its dual $\overset{*}{F}$ are the (unique up to scaling) simple non-null spacelike and timelike bivectors in the range of τ, respectively and $\alpha, \beta \in R$.

Class C

This class can have rank two or three. Eq.(2.1) have a unique (up to scaling) solution say, u and also $\dim S_p = 1$. The form of Riemann tensor at p will be

$$R_{abcd} = \sum_{i,j=1}^{3} \alpha_{ij} F^i{}_{ab} F^j{}_{cd}, \qquad (2.3)$$

here $\alpha_{ij} \in R$ for all i, j and $F^i{}_{ab} k^b = 0$ for each of the bivectors F^i which span the range of τ.

Class D

In this class the rank of the curvature matrix is one. There exist exactly two independent solutions k, u of Eq.(2.1) so that $\dim S_p = 2$. The Riemann tensor at p takes the form

$$R_{abcd} = \lambda\, F_{ab} F_{cd}, \qquad (2.4)$$

where $\lambda \in R$ and F is simple bivector with blade orthogonal to k and u. For detail (Hall, 2004).

Class O

Here rank of the Riemann curvature matrix is zero, then this class is called class O. In this class $R_{abcd} = 0$ and clearly $\dim S_p = 4$.

Class A

The Riemann tensor is said to be of class A at p if it is not of class B, C, D or O. Here always $\dim S_p = 0$. More detail of the (CCS) for all the above classes A, B, D, C and O can be seen in (Hall and da Costa, 1991).

3. RESULTS AND DISCUSSION

Consider the spatially homogeneous rotating Som-Roy Chaudhary symmetric spacetime with line element (Krori, et al., 1988).

$$ds^2 = -dt^2 + dr^2 + r^2(1 - r^2)d\theta^2 + dz^2 + 2r^2 dt d\theta. \qquad (3.1)$$

The above spacetime admits minimal three linearly independent Killing vector fields which are

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}. \qquad (3.2)$$

It can be easily shown that there exist four non-zero components of the Riemann tensor given by

$R_{0101} = 1 = \alpha_1$, $R_{0202} = R_{0112} = r^2 = \alpha_2$,

$R_{1212} = r^4 + 3r^2 = \alpha_3$. The 6×6 symmetric matrix form of the curvature tensor with components R_{abcd} at p is (Shabbir and Ramzan, 2008).

$$R_{abcd} = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.3)$$

For calculating CCS, we will consider Riemann tensor components as $R^a{}_{bcd}$. The spacetime metric may admit proper CCS when the rank of the 6×6 Riemann matrix is less than or equal to three (Hall and da costa, 1991).The method of finding the rank followed which is given in reference (Shabbir and Ramzan, 2008).Thus following two are the only surviving possibilities when the rank of Riemann matrix is less than or equal to three:

I. Rank 3: $\alpha_1 \neq 0, \quad \alpha_2 \neq 0, \quad \alpha_3 \neq 0,$

II. Rank 0: $\alpha_1 = \alpha_2 = \alpha_3 = 0.$

4. BRIEF DERIVATION OF CURVATURE COLLINEATIONS

Case I

In this case we have $\alpha_1 \neq 0,\ \ \alpha_2 \neq 0,\ \ \alpha_3 \neq 0,$ and the rank of the 6×6 Riemann matrix is 3 and there exists unique no where zero spacelike vector field $z_a = z_{,a}$ which is covariantly constant satisfying $z_{a;b} = 0$. Now the Ricci identity gives $R^a{}_{bcd}\, z_a = 0$. According to the above constraints ,the line element (3.1) does not change i.e

$$ds^2 = -dt^2 + dr^2 + r^2(1-r^2)d\theta^2 + dz^2 + 2r^2 dt d\theta. \tag{4.1}$$

The space-time (4.1) is 1+3 decomposable and clearly belongs to the curvature class C. The CCS in this case take the form (Hall and da. Costa,1991) $X = j(\text{z})\dfrac{\partial}{\partial z} + X'$, where $j(\text{z})$ is an arbitrary function of z and X' is the homothetic vector field in the induced geometry on each of the three dimensional sub manifolds of constant z. For the completion of this case, it is required to find the homothetic vector fields in the induced geometry. The non-zero components of the induced metric are

$$g_{00} = -1\ ,\ g_{11} = 1\ ,\ g_{22} = \text{r}^2(1-r^2),\ g_{02} = 2r^2.$$

For X' to be homothetic vector field must satisfy (Shabbir and Ramzan,2007)

$$\underset{X'}{L}\, g_{ab} = 2\alpha g_{ab}, \text{ where } \alpha \in R. \tag{4.2}$$

Expansion of the equation (4.2) results in the six non-linear coupled partial differential equations:

$$X^0_{,0} - r^2 X^2_{,0} = \alpha. \tag{4.3}$$

$$X^1_{,0} - X^0_{,1} + r^2 X^2_{,1} = 0. \tag{4.4}$$

$$2\text{r}\text{X}^1 + r^2\, \text{X}^0_{,0} + r^2(1-r^2)\,\text{X}^2_{,0} - \text{X}^0_{,2} + r^2\, \text{X}^2_{,2} = 2\alpha r^2 \tag{4.5}$$

$$X^1{}_{,1} = \alpha. \tag{4.6}$$

$$r^2\, \text{X}^0_{,1} + r^2(1-r^2)\,\text{X}^2_{,1} + \text{X}^1_{,2} = 0. \tag{4.7}$$

$$r(1-2r^2)\,\text{X}^1 + r^2\, \text{X}^0_{,2} + r^2(1-r^2)\,\text{X}^2_{,2} = \alpha r^2(1-r^2). \tag{4.8}$$

From equation (4.6), we have

$$X^1 = \alpha r + E^1(t,\theta). \tag{4.9}$$

Now solving equations (4.3) and (4.4) simultaneously and then using (4.9), we get

$$X^2 = \frac{1}{2r} E^1{}_t(t,\theta) + E^2(\text{r},\theta) \quad \text{and}$$

$$X^0 = \alpha t + \frac{r}{2} E^1{}_t(t,\theta) + E^3(\text{r},\theta)$$

Therefore,

$$\left.\begin{aligned} X^0 &= \alpha t + \frac{r}{2} E^1{}_t(t,\theta) + \int r^2 E_r{}^2(r,\theta)\,\text{dr} + E^4(\theta),\\ X^1 &= \alpha r + E^1(t,\theta),\\ X^2 &= \frac{1}{2r} E^1{}_t(t,\theta) + E^2(\text{r},\theta). \end{aligned}\right\} \tag{4.10}$$

Now using (4.10) in the remaining equations and after some lengthy calculation, we have

$$\left.\begin{aligned} X^0 &= \alpha t + (r^7 - \frac{r^3}{3})\,\text{c}_1 - \text{r}\, E^6{}_\theta(\theta) + E^4(\theta),\\ X^1 &= \alpha r + tc_1 + E^6(\theta)\\ X^2 &= c_1 \operatorname{cosech} 2\,\text{r} + \frac{1}{2}\coth 2r E^6{}_\theta(\theta) + E^8(\theta). \end{aligned}\right\} \tag{4.11}$$

Now substituting (4.11) in equation (4.5) and after doing some calculation, one can have

$$\left.\begin{aligned} X^0 &= \alpha t + r(c_3 \sin\theta - \text{c}_4 \cos\theta) + c_5,\\ X^1 &= \alpha r + c_3\cos\theta + c_4 \sin\theta\\ X^2 &= -\alpha\theta + \frac{1}{r}(c_4\cos\theta - c_3\sin\theta) + c_2. \end{aligned}\right\} \tag{4.12}$$

Use of the above information given in (4.12) in equation (4.8) and straight forward calculation gives $\alpha = 0$. It means that homothetic vector fields in the induced geometry of three dimensions are the Killing vector fields which are

$$\left.\begin{aligned} X^0 &= r(c_3 \sin\theta - \text{c}_4 \cos\theta) + c_5,\\ X^1 &= c_3\cos\theta + c_4 \sin\theta\\ X^2 &= \frac{1}{r}(c_4\cos\theta - c_3\sin\theta) + c_2. \end{aligned}\right\} \tag{4.13}$$

Hence curvature collineations of the above space-time become

$$\left.\begin{aligned} X^0 &= r(c_3 \sin\theta - c_4\cos\theta) + c_5, \\ X^1 &= c_3 \cos\theta + c_4 \sin\theta, \\ X^2 &= \frac{1}{r}(c_4 \cos\theta - c_3 \sin\theta) + c_2, \\ X^3 &= j(\mathrm{z}) \frac{\partial}{\partial z}. \end{aligned}\right\} \quad (4.14)$$

The proper CCS excluding Killing vector fields are

$$(0, 0, 0, j(z)), \quad (4.15)$$

where $j(\mathrm{z})$ is an arbitrary function. from the litraure it can be seen that proper curvature collineations in this case form an infinite dimensional vector space.

Case II

In this case all the components of the Riemann tensor are zero and the spacetime becomes Minkowski. The rank of 6×6 Riemann matrix becomes zero. This is the trivail case and belongs to the class O. The Ricci identity given in (2.1) is trivially satisfied by the vector fields $t, r, \theta, z.$ So every vector field is CC. The CCS in this case form an infinite dimensional vector space (Shabbir, *et al.*, 2003) .

5, <u>CONCLUSION</u>

In this paper a study of Som-Roy Chaudhary symmetric space-time which is basically a spatially homogeneous rotating space-time, according to their proper curvature collineations is presented. An approach is adopted to study the above space-time by using the rank of the 6×6 Riemann matrix and also using the well- known theorem which gives the conditions where proper curvature collineations exist. The above investigation reveals the following results:

(i) The case when the rank of the 6×6 Riemann matrix is three and there exists a unique nowhere zero independent spacelike vector field which is a solution of equation (2.1) and is covariantly constant. This is the space-time (3.1) and it admits proper CCS which form an infinite dimensional vector space.

(ii) In case II the space-time becomes Minkowski and every vector field is trivially curvature collineation.

<u>REFERENCES:</u>

Bokhari, A. H, A. Qadir and A. R. Kashif, (2003). A complete classification of cylindrical symmetric static merices, Gen. Rel. Grav. 35, 1059-1076.

Hall, G. S., J. da Costa,(1991). Curvature collineations in general relativity II, J. Math. Physics, 32, 2854-2862.

Katzin, G. H., J. Levine, W. R Davis, (1969). Curvature collineations: a fundamental symmetry property of the spacetimes of general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor, J. Math. Physics, 10, 617-629.

Krori, K. D, P. Borgohain, P. K. Kar, and D. Das (Kar), (1988). Exact scalar and spinor solutions in some rotating universes, J. Math. Phys. 29, 1645-1649.

Shabbir, G and M. Ramzan, (2007). Classification of cylindrically symmetric static space-times according to their proper homothetic vector fields, Applied Sciences, 9, 148-49.

Shabbir, G., A. H Bokhari, A. R Kashif, (2003). Proper curvature collineations in cylindrically symmetric static space-times, NUOVO CIMENTO B, 118, 873-886.

Stephani, H., D. Kramer, M. A. H MacCallum, C. Hoenselears and E. Herlt, (2003). Exact Solutions of Einstein’s Field Equations, Cambridge University Press.