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# Invariant Solution of Viscoelastic Fluid flow in Channels without Porous Media Attached with Constitutive Model of Oldroyd–B Resolved By Symmetry Method

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**Abstract:** This paper concerned with Viscoelastic fluid Flow in Channels without Porous media.. The related problem is equipped by using the model of Darcy-Brinkman and Constitutive model of Oldroyd–B . The research is arranged through analytical solutions of the governing system of equations, occurring in the investigation of viscoelastic flow in channels. The invariant solution is recognized by Lie group method. For finding exact invariant solutions, the PDEs is transferred in to ODEs corresponding to the symmetry group

**Keywords**: Viscoelastic Flow without Porous channels, Invariant Solutions of Darcy-Brinkman model with Oldroyd–B constitutive model by symmetry method

## 1. INTRODUVTION

The necessary and main branch of fluid mechanics is troubled with fluids that are frequently referred as complex, in recognition of the truth that these materials exhibits a great deal intricate behaviour. There are different analytical techniques to find the solution of PDEs in fluid mechanics. However, every one technique has their restrictions in applications. The Lie group theory is necessary approach in this observation. To find analytical solutions of behavior PDEs have been used pertaining to investigate in this field. Due to difficulty of fluids, there are many models relating the properties, but not all, of non-Newtonian fluids. Thus here the research and study of these fluids is hard, it is vital from a practical point of observation and understanding the non-Newtonian fluids itself. The fluids Flows related with some vital research are organized by approach of Abel-Malek *et al* (2002), Ariel *et al.* (2006), Bird *et al.* (1981, 1983, 1987), Chen, *et al.* (2006), Fetecau and Fetecau (2005. 2006), Rajagopal, (1984, 1985), Oldroyd (1958), Owens and Phillips (2002), van Os and Phillips (2004), Taha (2009, 2010) and Wafo-Soh (2005), and others. In a viscoelastic flow, the pressure tensor relative to the deformation gradient history; researched by Moran, and Gaggioli. (1968). Phenomena related with Viscoelastic flows have been studied and investigated by Taha (2009, 2010) and Larson.(1999).

There are different analytical techniques to solve Partial differential equations occurring in fluid mechanics.. Recently a developed analysis method is symmetry method or Lie group method, this method has been successfully applied by various researchers. In this paper, our aim is to solve PDE's system by Lie-Group theory using lie-point symmetries method and obtained their exact solutions. Lie-Group theory of ODE's and PDE's as a methodical branch constructed from hard work of the excellent mathematician Lie (1842 to 1899) and developed by Bluman and Kumei (1989), Olver, (1986), Ibragimov, (1999) and others researchers. The dynamics of mainly physical methods is explained by differential equations and for acquiring exact solutions of such equations, Lie group theory supplies powerful tools.

Section 2 concerns with the problem formulation. Section 3 associated with invariant results of viscoelastic fluid flow without porous space in channel, section 3.1 connected with Solution of Study State, segment 3.2 contains Lie Group method of PDEs (13-i) and (13-iii). Sections 3.3 is concerned with Lie-point generators, segment 3.4 connected with results concerned with invariant solution of PDE's (13-i) and (13-iii) related to the operator $X_1 - \alpha X_3$, section 3.5 connected with results of the equation (13-ii), section 3.6 concerned with result corresponding to the symmetry $X_1$. Segment 4 includes with conclusions.

## 2. FORMULATIONS OF PROBLEM

Suppose viscoelastic fluid flow which is the incompressible laminar in a porous channel. The system of equations relative to flow comprises of the conservation of mass and momentum transport combined with the constitutive model of Oldroyd–B. The viscoelastic fluid flows through porous medium are supposed to exist homogeneous and isotropic. Using model Darcy-Brinkman, the equation of momentum can be modelled and continuity equation and momentum equation in the absence of body force may be written under the following equations:

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$$\nabla.\bar{u} = 0 \qquad (1) \qquad\qquad \frac{\rho}{\varepsilon}\frac{\partial \bar{u}}{\partial t} = \frac{1}{\varepsilon}\nabla([2\mu_2 \underline{\underline{d}}] + \tau) - \nabla p - \rho\,\bar{u}.\nabla\bar{u} - \frac{\mu}{K}\bar{u} \qquad (2)$$

The constitutive equation explains the stresses in the viscoelastic flow can be expressed as:

$$\lambda\frac{\partial \tau}{\partial t} = [2\mu_1 \underline{\underline{d}}] - \tau - \lambda\{\,\bar{u}\,.\nabla\,\tau - \nabla\bar{u}\,.\tau - (\nabla\bar{u})^T\,.\tau\} \qquad (3)$$

Where velocity vector field is $\bar{u}$, extra stress tensor is $\tau$, rate of strain tensor is indicated by $\underline{\underline{d}}$, $\nabla$ is signified the operator of spatial differential, *p* is pressure (per unit density) of the isotropic fluid and *t* is related with the time. The $\square_1$ is indicated the viscoelastic solute and $\square_2$ is signified the Newtonian solvent viscosities, ρ is the fluid density, relaxation time of the viscoelastic fluid is denoted by *λ* and intrinsic permeability of the porous media *is denoted by K*, the coefficient of Forchheimer is c. $\square = \square_1 + \square_2$ is the total viscosity □□□flow and is constant. In equation (2), the co–efficient tensor of acceleration is supposed to be $1/\varepsilon$ and porosity of porous medium is $\varepsilon$ .The derived equations preside over the unsteady viscoelastic unidirectional fluid flow during porous media accepting constitutive model of Oldroyd–B. The velocity field is $\bar{u} = (0,\, v(t, y), 0)$ for unidirectional fluid flow, hence the report of velocity mechanically provides delight to the incompressibility condition. The derivation of such equations supposing pressure gradient as a constant and can be stated as: under:

$$\frac{\rho}{\varepsilon}\frac{\partial v}{\partial t} = \frac{\mu_2}{\varepsilon}\frac{\partial^2 v}{\partial y^2} + \frac{1}{\varepsilon}\frac{\partial \tau_{12}}{\partial y} - \frac{\partial p}{\partial x} - \frac{\mu}{K}v \quad (i) \qquad \lambda\frac{\partial \tau_{11}}{\partial t} = 2\lambda\,\tau_{12}\frac{\partial v}{\partial y} - \tau_{11} \quad (ii)$$

$$\lambda\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial v}{\partial y} - \tau_{12} \quad (iii) \qquad (4)$$

Where $v(t, y)$ is the component of velocity in direction of axial and $\tau_{11}(t, y)$, $\tau_{12}(t, y)$ and $\tau_{22}(t, y)$ are the components of stress tensor in direction of axial, direction of shear and direction of transversal. As transversal direction is denoted by *y*, where $\tau_{12}$.second normal stress which is vanish i-e $\tau_{12} = 0$.

By introducing the non dimensional variables, the dimensionless equations system is expressed in the following form $v = v^* Vc$, $\tau = \frac{\mu Vc}{L}\tau^*$, $y = L\,y^*$, $K = K^*$ and $t = \frac{L}{Vc}t^*$

Along with material parameters $\lambda = \frac{L}{Vc}\lambda^*$, $\mu_1 = \mu\,\mu_1^*$, $\mu_2 = \mu\,\mu_2^*$

where $v^*$ is dimensionless velocity, $\square^*$ is dimensionless stress and $y^*$ is non-dimensional transversal coordinates and $t^*$ is dimensionless time and the non-dimensional modified permeability is $K^*$.As, characteristic length is *L* as half width of the channel and the characteristic velocity *is Vc* supposed as reference axial velocity $Vc = \varepsilon R^2\left(-\frac{\partial p}{\partial z}\right)\Big/\mu$, then after substituting the dimensionless variables, the equations which are non-dimensional form turn into

$$\mathrm{Re}\frac{\partial v}{\partial t} = 1 + \mu_2\frac{\partial^2 v}{\partial y^2} + \frac{\partial \tau_{12}}{\partial y} - \frac{1}{Da}v \quad (i) \qquad We\frac{\partial \tau_{11}}{\partial t} = 2We\,\tau_{12}\frac{\partial v}{\partial y} - \tau_{11} \quad (ii)$$

$$We\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial v}{\partial y} - \tau_{12} \quad (iii) \qquad (5)$$

Where $\mathrm{Re}$, $We$ and $Da$ re the non-dimension Reynolds number, non-dimension Weissenberg number and non-dimension Darcy's number. As $\mathrm{Re} = \rho L Vc\big/\mu$, $We = \lambda Vc\big/L$ and $Da = K^*\big/\varepsilon L^2$ are respectively.

## 3. VISCOELASTIC FLUID FLOW IN CHANNELS WITHOUT POROUS SPACE

As Darcy's number $Da$ approaches to infinity or the last term vanishes, then the differential equations system (5) is called flow of viscoelastic fluid in channels without porous medium and is written as:

$$\mathrm{Re}\frac{\partial v}{\partial t} = 1 + \mu_2\frac{\partial^2 v}{\partial y^2} + \frac{\partial \tau_{12}}{\partial y} \quad (i) \qquad We\frac{\partial \tau_{11}}{\partial t} = 2We\,\tau_{12}\frac{\partial v}{\partial y} - \tau_{11} \quad (ii)$$

$$We\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial v}{\partial y} - \tau_{12} \quad (iii) \qquad (6)$$

To complete the well posed problem requirement, it is necessary to set initial and boundary conditions. So initial conditions are take as:

$v\,(0, y) = 0,\ \tau_{11}(0, y)=0,.\ \tau_{12}(0, y)=0,.$ when t > 0 (7)

and boundary conditions are given as:

$v$(t, –1) = 0 and $v$(t, 1) = 0, when $-1\le y \le 1$ (8)

**3.1 Exact solutions of equations system (6 to 8) of viscoelastic flow through channels without porous media**

i. **Solution for non-homogeneous equations system**

For result of the study state, the equations of non–homogeneous system are able to solve by resources of a transform of dependent variables, so for this suppose

$$v\,(t,\, y) = u_1(t,\, r)+\psi_1(y),\quad \tau_{11}(t, r) = u_2(t,\, y)+\psi_2(y)\quad \text{and}\quad \tau_{12}\,(t,\, r\,) = u_3\,(t,\, y)+\psi_3(y) \tag{9}$$

Putting these values in Equation (6), and separating the like terms of two independent variables which gives the two systems of equations which are given as

$$\left.\begin{array}{ll} 1+\mu_2\,\psi_1''(y)+\psi_3'(y)=0 \quad (i) & 2We\,\psi_3(y)\,\psi_1'(y)-\psi_2(y)=0 \quad (ii) \\ \mu_1\psi_1'(y)-\psi_3(y)=0 \quad (iii) & \end{array}\right\} \tag{10}$$

Subject to boundary conditions:

$$\psi_1(-1)=0,\quad \psi_1(1)=0 \ \text{and}\quad \mu_1+\mu_2=1 \tag{11}$$

After solving and integrate (10) and applying the boundary conditions (11), result of above system (6) admit the steady–state solutions as below:

$$\left.\psi_1(y)=\frac{1}{2}\,(1-y^2)\qquad \psi_2(y)=2We\,\mu_1\;y^2\qquad \psi_3(y)=-\mu_1\,y\right\} \tag{12}$$

and second system which is PDE's is given as

$$\begin{array}{ll} \mathrm{Re}\,\dfrac{\partial u_1}{\partial t} = \mu_2\,\dfrac{\partial^2 u_1}{\partial y^2}+\dfrac{\partial u_3}{\partial y} \quad (i) & We\,\dfrac{\partial u_2}{\partial t}+u_2 = 2We\,(u_3-\mu_1\,y)\,\dfrac{\partial u_1}{\partial y}-2We\,y\,u_3 \quad (ii) \\ We\,\dfrac{\partial u_3}{\partial t}+u_3=\mu_1\,\dfrac{\partial u_1}{\partial y} \quad (iii) & \end{array} \tag{13}$$

Subject to initial and boundary conditions is given as:

$u_1(t,-1)=0,$ (14- i) and $u_1(t,1)=0,.$ (14-ii) when t > 0

$u_1(0,y)=-\dfrac{1}{2}\,(1-y^2),.$ (14-iii) $u_2(0,y)=-2We\,\mu_1\;y^2,$ (14-iv) $u_3(0,y)=\mu_1\,y$ (14-v) when $-1\le y\le 1$

**3.2 Lie Group Method for Solving the PDEs system**

Group method or symmetry method is powerful method in finding exact solutions of differential equations. As Lie group of the equation is identified, it can be applied in the search of transformations. Symmetry will decrease the equation in easy form. As derivatives of equations 13-i and 13-iii are linked each other. So Lie group method for obtaining the Lie point symmetries of these equations is introduced. The generator

$$X=\phi(t,y,u_1,u_3)\frac{\partial}{\partial t}+\xi(t,y,u_1,u_3)\frac{\partial}{\partial y}+\eta^1(t,y,u_1,u_3)\frac{\partial}{\partial u_1}+\eta^2(t,y,u_1,u_3)\frac{\partial}{\partial u_3} \tag{15}$$

is the Lie point symmetry generator for governed PDEs system (13-i) and (13-iii) if,

$$X^{[2]}(\mu_2\;u_{1yy}+u_{3y}-\mathrm{Re}\;u_{1t})\Big|_{(13-i\,\&\,iii)}=0\qquad X^{[1]}(\mu_1\,u_{1y}-We\;u_{3\,t}-u_3)\Big|_{(13-i\,\&\,iii)}=0 \tag{16}$$

where first and second extended infinitesimal generator of $X$ are

$$X^{[1]}=X+\eta_t^{1[1]}\frac{\partial}{\partial u_{1t}}+\eta_y^{1[1]}\frac{\partial}{\partial u_{1y}}+\eta_t^{2[1]}\frac{\partial}{\partial u_{3\,t}}+\eta_y^{2[1]}\frac{\partial}{\partial u_{3y}}\qquad X^{[2]}=X^{[1]}+\eta_{yy}^{1[2]}\frac{\partial}{\partial u_{1yy}} \tag{17}$$

In which \

$$\eta_t^{1[1]} = D_t\eta^1 - u_{1t}\, D_t\phi - u_{1y}\, D_t\xi\,, \qquad \eta_y^{1[1]} = D_y\eta^1 - u_{1t}\, D_y\phi - u_{1y}\, D_y\xi, \qquad \eta_t^{2[1]} = D_t\eta^2 - u_{3t}\, D_t\phi - u_{3y}\, D_t\xi\,;$$

$$\eta_y^{2[1]} = D_y\eta^2 - u_{3t}\, D_y\phi - u_{3y}\, D_y\xi\,; \qquad \eta_{yy}^{1[2]} = D_y\eta_y^{[1]} - u_{1ty}\, D_y\phi - u_{1yy}\, D_y\xi\,. \tag{18}$$

where $D_{x_i}$ is the total derivative operator given as:

$$D_t = \frac{\partial}{\partial t} + u_{1t}\frac{\partial}{\partial u_1} + u_{3t}\frac{\partial}{\partial u_3} + u_{1tt}\frac{\partial}{\partial u_{1t}} + u_{3tt}\frac{\partial}{\partial u_{3t}} + u_{1ty}\frac{\partial}{\partial u_{1y}} + ...,\ D_y = \frac{\partial}{\partial y} + u_{1y}\frac{\partial}{\partial u_1} + u_{3y}\frac{\partial}{\partial u_3} + u_{1yy}\frac{\partial}{\partial u_{1y}} + u_{3yy}\frac{\partial}{\partial u_{3y}} + u_{1ty}\frac{\partial}{\partial u_{1t}} + ..., \tag{19}$$

According to Lie's theory, in the operator *X*, the unknown functions $\phi$, $\xi$, $\eta^1$ and $\eta^2$ are taken independent of the derivatives of the primitive variables $u_1$ and $u_3$. Hence

$$X^{[2]}(\mu_2\, u_{1yy} + u_{3y} - \mathrm{Re}\, u_{1t})\Big|_{(13-i\,\&\,iii)} = 0 \qquad \Rightarrow -\mathrm{Re}\, \eta_t^{1[1]} + \eta_y^{2[1]} + \mu_2\, \eta_{yy}^{1[2]}\Big|_{(13-i\,\&\,iii)} = 0\Big\} \tag{20}$$

Similarly $X^{[1]}(\mu_1 u_{1y} - We\, u_{3t} - u_3)\Big|_{(13-i\,\&\,iii)} = 0 \Rightarrow -\eta^2 + \mu_1\, \eta_y^{1[1]} - We\, \eta_t^{2[1]}\Big|_{(13-i\,\&\,iii)} = 0\Big\}$ (21)

Where $X^{[1]}$, $X^{[2]}$ and ($\eta_t^{1[1]}$, $\eta_r^{1[1]}$, $\eta_{rr}^{1[2]}$, $\eta_t^{2[1]}$, $\eta_r^{2[1]}$) are described in the relations (18 and 19). As unknown functions $\phi$, $\xi$, $\eta^1$ and $\eta^2$ are independent for the differentials of $v_1$ and $v_3$.Therefore unscrambling with respect. to powers of the differentials of $v_1$ and $v_3$ guides to the two basic over concluded PDE's systems and after solving these two over determined systems of linear PDE's, solution of the two over resolved systems gives the values of $\phi$, $\xi$, $\eta^1$ and $\eta^2$ functions as

### 3.3 Lie-point symmetries of the partial differential equations (13-i) & (13-iii)

Solution of the two over resolved systems of (21) and (23) gives rise to the values of the functions $\phi$, $\xi$, $\eta^1$ and $\eta^2$ are given as:

$$\phi = c_1\,, \quad \xi = c_2\,, \qquad \eta^1 = c_3\, u_1 + g_1(t, y) \quad \text{and} \qquad \eta^2 = c_3\, u_3 + g_2(t, y) \tag{24}$$

Where $g_1(t, y)$ and $g_2(t, y)$ are the arbitrary functions of the equations

$$\mathrm{Re}\ g_{1t} = \mu_2\ g_{1yy} + g_{2y} \quad \text{and} \quad We\ g_{2t} = \mu_1\, g_{1y} - g_2 \tag{25}$$

Hence $c_i$ are arbitrary constants. Thus the symmetry Lie algebra of the partial differential equations (13-i) and (13-iii) is three–dimensional and defined by the following generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = u_1\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_3}, \text{and } X_m = g_1(t, y)\frac{\partial}{\partial u_1} + g_2(t, y)\frac{\partial}{\partial u_3} \tag{26}$$

where *m* are non-negative numbers. (27)

### 3.4 Invariant solutions of the PDE's (13-i) & (13-iii) corresponding to the generator *X = X₁ - α X₃*

From given generator (15), the invariant solutions corresponding to *X*, are obtained by solving the characteristic system

$$\frac{dt}{\phi} = \frac{dy}{\xi} = \frac{du_1}{\eta^1} = \frac{du_3}{\eta^2} \tag{28}$$

Hence only those operators are used for solving the problem which represents meaningful physical solutions of the problem consisting with the governing partial differential equations (13-i) and (13-iii). This method is used to reduce the PDE's (13-i) and (13-iii) to solvable form. As

$$X = X_1 - \alpha\, X_3 = \frac{\partial}{\partial t} - \alpha\, u_1\frac{\partial}{\partial u_1} - \alpha\, u_3\frac{\partial}{\partial u_3} \qquad \Rightarrow \frac{dt}{1} = \frac{-du_1}{\alpha\, u_1} = \frac{-du_3}{\alpha\, u_3}$$

The invariant results admitted by the generator *X* are given as

$$u_1(t, y) = e^{-\alpha t}\, \phi_1(y)\,, \qquad u_3(t, y) = e^{-\alpha t}\, \phi_3(y)\Big\} \tag{29}$$

Replacement the above relations (29) into governing equations (13-i) and (13-iii) represents ordinary differentials equations of functions $\phi_1$(y) and $\phi_3$(y).

$$\mu_2\,\phi_1''(y)+\phi_3'(y)+\alpha\,\mathrm{Re}\;\phi_1(y)=0 \quad (i) \qquad \mu_1\,\phi_1'(y)-(1-\alpha\,We)\phi_3(y)=0 \quad (ii) \tag{30}$$

Where prime stands for derivatives of *y*, solving the above system, substituting the value of $\phi_3(y)$ from (30-i) in to (30-ii) the following equation is obtained

$$\phi_1''(y)+\lambda^2\,\phi_1(y)=0 \tag{31}$$

Subject to boundary conditions:

$$\phi_1(-1)=0,\quad \phi_1(1)=0 \tag{32}$$

where $\lambda^2=\dfrac{\alpha\,\mathrm{Re}\,(1-\alpha\,We)}{(1-\mu_2\,\alpha\,We)}$ and $\mu_1+\mu_2=1$

Solution of equation (31) is given as:

$$\phi_1(y)=c_1\,Cos\,\lambda\,y+c_2\,Sin\lambda\,y \tag{33}$$

Applying the boundary conditions from equation (32), we find that $c_2=0$ and where $c_1\neq 0$ because it becomes trivial solution, so $c_1\,Cos\,\lambda=0 \Rightarrow \lambda=Cos^{-1}0,\ \Rightarrow \lambda_n=(\frac{2n-1}{2})\pi$ where n are non-negative integers.

Substituting the value of $\lambda_n$ into (33) and applying the super position principle, then it takes the form

Solution of equation (31) is given as:

$$\phi_1(y)=\sum_{n=1}^{\infty}A_n\;Cos\lambda_n y \tag{34}$$

So $\phi_3(y)$ takes the form

$$\phi_3(y)=\frac{\mu_1\,\lambda}{(1-\alpha\,We)}(c_2\,Cos\,\lambda\,y-c_1\,Sin\lambda\,y) \tag{35}$$

where $A_n$ are different roots and $\lambda_n=(\frac{2n-1}{2})\pi$

Putting the values of $\phi_1(y)$ and $\phi_3(y)$ in to equation (29), then it takes the form

$$u_1(t,y)=\sum_{n=1}^{\infty}A_n\,e^{-\alpha t}\,Cos\lambda_n\,y \text{ and } \qquad u_3(t,y)=\sum_{n=1}^{\infty}\frac{-\mu_1\,\lambda_n}{(1-\alpha\,We)}A_n\,e^{-\alpha t}\,Sin\lambda_n\,y \tag{36}$$

As the equation (31) is the combination of two equations (30-i & 30-ii), which have same boundary conditions, so for functions of time, we consider

$$u_1(t,y)=\sum_{n=1}^{\infty}f_1(t)\,Cos\lambda_n y \quad\text{and}\quad u_3(t,y)=\sum_{n=1}^{\infty}f_2(t)\,Sin\lambda_n y \tag{37}$$

Substituting the above relations (37) into governing equations (13-i) and (13-iii), which represent the following ordinary differentials equations of function of time

$$\mathrm{Re}\,f_1'(t)+\mu_2\,\lambda_n^{\,2}f_1(t)-\lambda_n\,f_2(t)=0 \quad (i) \text{ and } \quad We\,f_2'(t)+f_2(t)+\mu_1\,\lambda_n\,f_1(t)=0 \quad (ii) \tag{38}$$

After solving the above system of equations (38-i & ii), we obtain

$$f_1(t)=A_{n_1}\,e^{-\alpha_1 t}+A_{n_2}\,e^{-\alpha_2 t} \quad (i) \text{ and } \quad f_2(t)=-\mu_1\,\lambda_n\left(\frac{A_{n_1}}{(1-\alpha_1 We)}e^{-\alpha_1 t}+\frac{A_{n_2}}{(1-\alpha_2 We)}e^{-\alpha_2 t}\right) \quad (ii) \tag{39}$$

where $A_{n_1}$ & $A_{n_2}$ are constants with different roots and

$$\alpha_1=\frac{1}{2}\left(\frac{1}{We}+\frac{\mu_2\,\lambda_n^{\,2}}{\mathrm{Re}}\right)+\frac{1}{2}\sqrt{\left(\frac{1}{We}+\frac{\mu_2\,\lambda_n^{\,2}}{\mathrm{Re}}\right)^2-\frac{4\lambda_n^{\,2}}{\mathrm{Re}We}} \quad\text{and}\quad \alpha_2=\frac{1}{2}\left(\frac{1}{We}+\frac{\mu_2\,\lambda_n^{\,2}}{\mathrm{Re}}\right)-\frac{1}{2}\sqrt{\left(\frac{1}{We}+\frac{\mu_2\,\lambda_n^{\,2}}{\mathrm{Re}}\right)^2-\frac{4\lambda_n^{\,2}}{\mathrm{Re}We}}$$

where $u_1(t,y)$ and $u_3(t,y)$ of equation (37) turns over of the form as

$$u_1(t,y)=\sum_{n=1}^{\infty}(A_{n_1}\,e^{-\alpha_1 t}+A_{n_2}\,e^{-\alpha_2 t})Cos\lambda_n\,y \quad\text{and}\quad u_3(t,y)=\sum_{n=1}^{\infty}-\mu_1\,\lambda_n\left(\frac{A_{n_1}}{(1-\alpha_1 We)}e^{-\alpha_1 t}+\frac{A_{n_2}}{(1-\alpha_2 We)}e^{-\alpha_2 t}\right)Sin\lambda_n\,y \tag{40}$$

In order to satisfy the initial condition (14-iii & 14-v), we have,

$$A_{n_1} = \frac{2(-1)^{n-1}\alpha_2(1-\alpha_1 We)}{\lambda_n^{\,3}(\alpha_1-\alpha_2)} \quad \text{and} \quad A_{n_2} = \frac{-2(-1)^{n-1}\alpha_1(1-\alpha_2 We)}{\lambda_n^{\,3}(\alpha_1-\alpha_2)}$$

Substituting the values of $A_{n_1}$ and $A_{n_2}$ into (40), then it takes the form

$$\left.\begin{aligned} u_1(t,y) &= \sum_{n=1}^{\infty}\frac{16(-1)^{n-1}}{(2n-1)^3\pi^3}\left(\frac{\alpha_2(1-\alpha_1 We)}{(\alpha_1-\alpha_2)}e^{-\alpha_1 t} - \frac{\alpha_1(1-\alpha_2 We)}{(\alpha_1-\alpha_2)}e^{-\alpha_2 t}\right)Cos\left(\frac{2n-1}{2}\right)\pi\, y \\ u_3(t,y) &= \sum_{n=1}^{\infty}\frac{8(-1)^{n}\mu_1}{(2n-1)^2\pi^2}\left(\frac{\alpha_2}{(\alpha_1-\alpha_2)}e^{-\alpha_1 t} - \frac{\alpha_1}{(\alpha_1-\alpha_2)}e^{-\alpha_2 t}\right)Sin\left(\frac{2n-1}{2}\right)\pi\, y \end{aligned}\right\} \tag{43}$$

**3.5 Solution of the partial differentials equations (13-ii).**

Now, take equation (13-ii), we have

$$We\frac{\partial u_2}{\partial t} + u_2 = 2We(u_3 - \mu_1\, y)\frac{\partial u_1}{\partial y} - 2We\, y\, u_3$$

After putting the values of $\frac{\partial u_1}{\partial y}$ and $u_3$ from (40) and after finding the C.F. and P. I., and for the $y$ as

$$y = \sum_{n=1}^{\infty} c_n\, Sin\lambda_n y \Rightarrow c_n = \frac{\int_{-1}^{1} y\, Sin\lambda_n\, y\, dy}{\int_{-1}^{1} Sin^2\lambda_n\, y\, dy} = \frac{2(-1)^{n-1}}{\lambda_n^{\,2}} \Rightarrow y = \sum_{n=1}^{\infty}\frac{2(-1)^{n-1}}{\lambda_n^{\,2}} Sin\lambda_n y$$

Substituting the values of y, $A_{n_1}$ and applying the initial condition (14-iv), i-e $u_2(0,y) = -2We\,\mu_1\, y^2$, then it gives, the final solution of $u_2(t,y)$ is obtained as

$$u_2(t,y) = \left( \sum_{n=1}^{\infty}\frac{8(2We\,\mu_1)^{\frac{1}{2}}}{(2n-1)^2\pi^2}\left( \begin{aligned} &\frac{\alpha_2^2(1-\alpha_1 We)\,(e^{-2\alpha_1 t} - e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_1 We)} + \frac{\alpha_1^2(1-\alpha_2 We)\,(e^{-2\alpha_2 t} - e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2(1-2\alpha_2 We)} \\ &-\frac{\alpha_1\alpha_2(2-\alpha_1 We-\alpha_2 We)\,(e^{-(\alpha_1+\alpha_2)t} - e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2(1-\alpha_1 We-\alpha_2 We)} \\ &+\frac{\alpha_2(2-\alpha_1 We)(e^{-\alpha_1 t} - e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)(1-\alpha_1 We)} - \frac{\alpha_1(2-\alpha_2 We)(e^{-\alpha_2 t} - e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)(1-\alpha_2 We)} \end{aligned} \right)^{\frac{1}{2}} Sin\left(\frac{2n-1}{2}\right)\pi\, y \right)^2 - 2We\,\mu_1\, y^2\, e^{\frac{-t}{We}} \tag{45}$$

Final solution of the problem (6) subject to initial and boundary conditions (7 & 8) after substituting the solutions (12, 43 & 45) into (9) is find as:

$$v(t,y) = \sum_{n=1}^{\infty}\frac{16(-1)^{n-1}}{(2n-1)^3\pi^3}\left(\frac{\alpha_2(1-\alpha_1 We)}{(\alpha_1-\alpha_2)}e^{-\alpha_1 t} - \frac{\alpha_1(1-\alpha_2 We)}{(\alpha_1-\alpha_2)}e^{-\alpha_2 t}\right)Cos\left(\frac{2n-1}{2}\right)\pi\, y + \frac{1}{2}(1-y^2) \tag{46-i}$$

$$\tau_{11}(t,\,y)=\left(\sum_{n=1}^{\infty}\frac{8(2We\,\mu_1)^{\frac{1}{2}}}{(2n-1)^2\pi^2}\left(\begin{array}{l}\dfrac{\alpha_2^2(1-\alpha_1 We)\,(e^{-2\alpha_1 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_1 We)}+\dfrac{\alpha_1^2(1-\alpha_2 We)\,(e^{-2\alpha_2 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2\,(1-2\alpha_2 We)}\\ -\dfrac{\alpha_1\alpha_2(2-\alpha_1 We-\alpha_2 We)\,(e^{-(\alpha_1+\alpha_2)t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)^2(1-\alpha_1 We-\alpha_2 We)}\\ +\dfrac{\alpha_2\,(2-\alpha_1 We)(e^{-\alpha_1 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)(1-\alpha_1 We)}-\dfrac{\alpha_1\,(2-\alpha_2 We)(e^{-\alpha_2 t}-e^{-\frac{t}{We}})}{(\alpha_1-\alpha_2)(1-\alpha_2 We)}\end{array}\right)^{\frac{1}{2}} Sin\,(\frac{2n-1}{2})\,\pi\, y\right)^2+2We\,\mu_1\,y^2\,(1-e^{\frac{-t}{We}})$$

(46-ii)

$$\tau_{12}(t,\,y)=\sum_{n=1}^{\infty}\frac{8(-1)^n\,\mu_1}{(2n-1)^2\pi^2}\left(\frac{\alpha_2}{(\alpha_1-\alpha_2)}e^{-\alpha_1 t}-\frac{\alpha_1}{(\alpha_1-\alpha_2)}e^{-\alpha_2 t}\right)Sin\,(\frac{2n-1}{2})\,\pi\,y-\mu_1\,y \qquad (46\text{-iii})$$

## 4 CONCLUSIONS

In this paper, we have solved the problem of system of three partial differential equations related with viscoelastic flow without porous media within channels attached with constitutive model of Oldroyd–B. As major purpose of our investigation is to solve the problem of PDEs system by applying Lie group technique successfully and to obtain the invariant solution of the problem occurring in the research of viscoelastic flow in channels without porous medium.. In this paper, the PDE’s system was changed into ODE’s system and then these equations are able to be solved and get invariant solution. By applying symmetry conditions, Lie-point symmetries have been attained and accepted to reach at the solutions. This method can provide some realistic insights attention in comprise of outcomes and may support for formative demanding results in some cases.. We trust that the solutions may be helpful for other personnel in this field. Our recommendations for the future work are developing and setting into practice other Algorithms.

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