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A Modified Algorithm for Reduction of Error in Combined Numerical Integration

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Abstract: In this paper, a new modified algorithm is derived for combined numerical integration. Numerical integration is used to evaluate definite integral which cannot be evaluated analytically. In order to obtain higher order accuracy in the solution, mostly higher order quadrature rules are used which also introduces rounding off errors. In order to mitigate such errors, it has been proposed to use a combination of lower order rules to improve the accuracy and reduce error. Several definite integrals have been approximated and the results have been compared with the existing rules and a rule proposed by (Md.Amanat Ullah, 2015). The rules from the closed quadrature family, namely, Weddle's rule, Boole's rule, Simpson's $\frac{3}{8}$ rule and Trapezoidal rule have also been used. It has been found that the new proposed modified algorithm attains improved order of accuracy in comparison of the existing rules and the rule of Md.Amanat Ullah for a fixed number of subintervals.

Keywords: Numerical integration, Trapezoidal rule, Simpson's 3/8 rule, Boole's rule and Weddle's rule.

1. INTRODUCTION

Definite integration in mathematics is one of the most basic and important concept. It has numerous applications in various fields such as physics and engineering. Definite integration is the approximation of numerical values that cannot be integrated analytically (Gordon KS,1998).Various numerical integration techniques such as Gauss Quadrature rules, Monte Carlo integration, Romberg integration and Newton-Cotes methods, are used for the evaluation of those functions, Newton-Cotes methods such as the Trapezium rule, Simpson's$\frac{1}{3}$rd rule, Simpson's$\frac{3}{8}$th rule, Boole's rule and Weddle's rule are special cases of 1st order, 2nd order, 3rd order, 4th order and 6th order polynomials respectively. The Trapezium rule has no limit on the number of subintervals. The number of subinterval for the Simpson's$\frac{1}{3}$rd rule must be even and for Simpson $\frac{3}{8}$rule, the number of subintervals must be a multiple of 3, for the Boole's rule the number of subintervals must be a multiple of 4 and for Weddle's rule the number must be a multiple of 6 (Felix O. Mettle1, et al., 2016). The primary goal of numerical integration is to provide alternatives to approximate definite integrals with finite integration boundaries; numerical integration is useful in approximating the integrals when only the discrete behavior of the integrand is known in a bounded range instead of the closed form integrand itself (Shaikh,et al., 2016).

Many researchers have already carried out an extensive research work in the field of numerical integration. The numerical integration formulae are described in the book (R.L. Burden, 2007) and many other authors.

Soomro et al., (2013) used the mid-point integration rule for nonlinear differential equations. Recently, Shaikh (2019) demonstrated the preference of quadrature rules over the polynomial collocation for solving the Fredholm integral equations of second kind. Dehghan et al. (2005, 2006) proposed improvements to open and closed Newton-Cote's quadrature rules by considering the width of interval as an additional parameter. Zhao and Li (2013) proposed new variants of the Newton-Cote's methods which has used even-order higher derivative (like: second, fourth, etc.) at the mid-point of each integration strip. Similar works are also due to (Burg, 2012; Burg and Degny, 2013). These rules also exhibit improved order of accuracy, particularly by two more units, than the original quadrature rules.

2. MATERIALS AND METHODS

2.1 Basic Numerical Integration

The Newton-Cotes formula in the form of polynomial is a frequently used interpolator feature in the interval $[a,b]$ and includes n points with $n-1$ order polynomial which goes through the abscissas x_i $(i=0,1,\dots,n)$ equally spaced. Approximating the region below the curve $y=f(x)$ from $x=a$ to $x=b$, the closed Newton-Cotes formula is used to interpolate Lagrange when polynomial are fitted (Gordon 1997). Letting $x_0=a,\ x_n=b$ and $h=\frac{b-a}{n}$

We have

$$f_n(x)=\sum_{i=0}^{n} f(x_i)L_i(x) \tag{1}$$

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Where n in $f_n(x)$ is the polynomial order approximating the function $y = f(x)$ given $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ and

$$L_i(x) = \prod_{k=0,\ k\neq i}^{n} \left(\frac{x - x_k}{x_i - x_k}\right), i = 0,1,2,\ldots,n \tag{2}$$

Where $L_i(x)$ is a weighting function comprising a product of $n-1$ terms with terms of $i = k$ omitted. Integrating $f(x)$ over $[a, b]$ and choosing $x_i = a + \frac{(b-a)i}{n}$ we have the Newton-Cotes rule;

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{3}$$

Where the weight w_i is defined by;

$$W_i = \int_a^b L_i(x)dx = \sum_{i=0}^{n}\left[\int_{x_0}^{x_1} f(x_i)\prod_{k=0,\ k\neq i}^{n}\left(\frac{x-x_k}{x_i-x_k}\right)dx,\ \ i = 0,1,2,\ldots,n\right] \tag{4}$$

When $n = 1,3,4$ and 6 we have followingSimple trapezoidal rule, Simpson $\frac{3}{8}$ rule, Boole's rule and the Weddle's rule respectively with local error term (Fiza Zafar, et al.,2014).

$$T = \frac{h}{2}[f(x_o) + f(x_1)] - \frac{h^3}{12}f^{(2)}(\xi) \tag{5}$$

$$S_{38} = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3}{80}h^5 f^{(4)}(\xi) \tag{6}$$

$$B = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8}{945}h^7 f^{(4)}(\xi) \tag{7}$$

$$W = \frac{3h}{10}[f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)] - \frac{h^7}{140}h^7 f^{(6)}(\xi) \tag{8}$$

for some $\xi \ \in [x_0, x_n]$.

2.2 Single and Multiple Integration Rules

Equations (5) to (8) are called single integration rule. Dividing the integration interval from a to b into a number of segments and applying the technique to each segment is one way to improve the accuracy of the above rules.It is the n possible to add the regions of individual segments to produce the integral for the whole interval. The resulting equ ations are called formulas for integration with multiple applications or composites (R L Burden, 2011).The Composite Trapezoidal rule(T^C), the composite Simpson's $\frac{3}{8}$ rule($S^C 38$), the composite Boole's rule (B^C) and the composite Weddle's rule (W^C) for multiple segments are given as under:

$$T^C = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n} f(x_i) + f(x_n)] \tag{9}$$

$$S^C 38 \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + 2f(x_6) + \cdots + f(x_n)] \tag{10}$$

$$B^C = \frac{2h}{458}\begin{bmatrix} 7\{f(x_0) + f(x_n)\} + 32\{f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{n-3}) + f(x_{n-1})\} \\ +12\{f(x_2) + f(x_6) + f(x_{10}) + \cdots + f(x_{n-6}) + f(x_{n-2})\} + \\ 14\{f(x_4) + f(x_8) + f(x_{12}) + \cdots + f(x_{n-8}) + f(x_{n-4})\} \end{bmatrix} \tag{11}$$

$$W^C = \frac{3h}{10}\begin{bmatrix} f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + \\ 5f(x_4) + f(x_5) + \cdots + f(x_{6n}) \end{bmatrix} \tag{12}$$

It has been observed that accuracy of the quadrature formulas has been found in the order- Simpson's three-eighth rule > Simpson's one-third rule > Boole's rule > Trapezoidal rule rule>Weddle's Rule (D.S. Ruhela, et al.,2014)**.**

2.3 Rule of Md. Amanat Ullah

Md.Amanat Ullah concluded that the Weddle's rule and then the Simpson's $\frac{1}{3}$ rule is more accurate among Trapezoidal rule and Simpson's $\frac{3}{8}$ rule but we know that number of subintervals must be divisible by 6 and 2 for the Weddle's rule and Simpson's $\frac{1}{3}$ rule respectively. But in some real situation the limits of an integral involving errors, in these cases his proposed rule, described below, can be implemented. Consider the number of subintervals of a given definite integral is n in which first n_1 subinterval is divisible by 6.Therefore, the rule of Weddle applies to the first$n_1$subintervals for the remaining of the $(n - n_1$) subintervals, consider next $n_2$subintervals are divisible by 2 so Simpson's $\frac{1}{3}$ rule can be used for these subinterval. From (n_1 -n_2) subintervals, Trapezoidal rule can be used for the last subintervals if n is an odd number.

Finally, to get the values of an integral, all the values acquired from these rules are added.

2.4 Modified Algorithm for Reduction of Error in Combined Numerical Integration

Consider the following definite integral to be approximated

$I = \int_{x_0}^{x_n} f(x)dx$ …………….. (A)

Let the total number of given subinterval is n $(n \geq 9)$

In which firstn_1 subintervals is divisible by *6* that is $n_1 = 6$.

To use present modified algorithm for reduction of error in combined numerical integration we have the following steps.

Step.1

Use single segment Weddle's rule (*W*) for each first n_1 subintervals in n

$$\mathbf{W} = \mathbf{I_1} = \int_{x_0}^{x_n} f(x)dx \approx \frac{3h}{10}[f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)]$$

Step.2

Let $r_1 = rem(n, 6)$, (No: of remaining subintervals), define n_2 the first subintervals in r_1 that is divisible by 4, so Boole's rule is applicable for the first n_2 subintervals

e.g. use single segment Boole's rule if n_2 comprises of one subinterval that is

$B{=}\boldsymbol{I_2} = \int_{x_0}^{x_n} f(x)dx \approx \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$

or use composite Boole's rule

$$B^C = \boldsymbol{I_2} = \int_{x_0}^{x_n} f(x)dx \approx \frac{2h}{458}\left[\begin{array}{c} 7\{f(x_0) + f(x_n)\} + 32\{f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{n-3}) + f(x_{n-1})\} \\ +12\{f(x_2) + f(x_6) + f(x_{10}) + \cdots + f(x_{n-6}) + f(x_{n-2})\} + \\ 14\{f(x_4) + f(x_8) + f(x_{12}) + \cdots + f(x_{n-8}) + f(x_{n-4})\} \end{array}\right]$$

Let $r_2 = rem(r_{1,}4)$ (No: of remaining subintervals) if $r_2 = 0$ Stop.

And add all computed values of I_1 and I_2 to get approximate value of definite integral (*A*).

Thus, $I = I_1 + I_2$.

Otherwise go to step.3

Step.3

Let n_3 be the first subintervals in r_2

If n_3 is divisible by 3 use Case-I

If n_3 is not divisible by 3 use Case-II

Case-I

If n_3 is divisible by 3 then use Simpson's $\frac{3}{8}$ rule (*S38*)

E.g. for single segment we have

$S38{=}I_3 = \int_{x_0}^{x_n} f(x)dx \approx \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$

Let $r_3 = rem(r_2, 3)$ (No: of remaining subintervals) if $r_3 = 0$ Stop.

And add all computed values of I_1, I_2 and I_3 to get approximate value of definite integral (*A*)

Thus, $I = I_1 + I_2 + I_3$

Otherwise use Trapezoidal rule (*T*) for remaining subintervals in r_3

E.g. if $r_3 = 1$ use Trapezoidal rule (*T*) for single segment

$T{=}I_4 = \int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}[f(x_0) + f(x_1)]$

And if $r_3 = 2$ use composite Trapezoidal rule (*T*)

$T{=}I_4 = \int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$

Finally add all computed values of I_1, I_2, I_3 and I_4 to get approximate value of definite integral (*A*)

Case-II

If first n_3 subintervals in r_2 is not divisible by *3*

Use Trapezoidal rule (*T*) for entire r_2 subintervals

E.g. if $r_2 = 1$ use Trapezoidal rule (*T*) for single segment

$T{=}I_3 = \int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}[f(x_0) + f(x_1)]$

And if $r_2 = 2$ use composite Trapezoidal rule (*T*)

$T{=}I_3 = \int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$

Finally add all computed values of $I_1, I_2$and I_3 to get approximate value of definite integral (*A*).

Thus, $I = I_1 + I_2 + I_3$

3. RESULTS AND DISCUSSION

In this paper, the approximate solutions are obtained using proposed modified algorithms (MA). The results are compared with the solutions obtained with the exiting existing methods (EM) and the method proposed by Md.Amanat Ullah (PMA). Several examples are taken to compare the accuracy of the solutions. All of the codes are implemented under MATLAB R2014a. Results on three examples are presented in the following tables. Absolute Error (Ab_error) and Percent Error (Per_Error) are computed using exact solution and approximate solution. The exact solution of the Example 1 in Table-1 and Table-2 for $n = 9{,}10{,}11$ and $n = 13{,}14{,}15$ respectively is taken from the research article of (Md.Amanat Ullah, 2015) while the exact solution of the Example 2 & 3 in Table-3 and Table-4 for $n = 9{,}10{,}11$ and $n = 13{,}14{,}15$ respectively is taken from the research article of (Francis,et al., 2017). It has been observed that the proposed modified method, for a particular *n*, attains improved order of accuracy in comparison of EMsand MPA for all examples.

Table-1, Example 1: $\int_0^1 \sqrt{1-x^2}\,dx = 0.7853981634$

Rules	For n=9	For n=10	For n=11
EM	0.774546345	0.776129582	0.777362076
Ab_Error	0.0108518184	0.00926858139	0.008036087
Per_Error	1.381696%	1.1801124362%	1.0231863753%
PMA	0.776456493	0.781754678	0.778798642
Ab_Error	0.0089416704	0.003643485	0.006599521
Per_Error	1.1384887%	0.463902969%	0.84027711%
MA	0.7802042676	0.782199413	0.778824026
Ab_Error	0.0051938958	0.00319875	0.006574137
Per_Error	0.661307%	0.40727755%	0.83704512%

In table-1, we found that the proposed modified algorithm (MA) depicts reduced errors in comparison of other methods. Using proposed modified algorithm, the error in the solution has been significantly reduced for *n*=9 and *n*=10 in comparison of other two methods. However, the accuracy has been improved slightly for *n*=11 in comparison of PMA. It has also been noticed that the MA fluctuates for larger *n*. Despite this fluctuation, the proposed method attains improved accuracy across other two methods.

Table-2, Example 1: $\int_0^1 \sqrt{1-x^2}\,dx = 0.7853981634$

Rules	For n=13	For n=14	For n=15
EM	0.779140619	0.779798012	0.780347853
Ab_Error	0.0062575444	0.005600151	0.005050310
Per_Error	0.796735%	0.71303337%	0.64302544%
PMA	0.780296297	0.78320648	0.781271184
Ab_Error	0.0051018663	0.002191683	0.004126979
Per_Error	0.649589%	0.279053746%	0.52546328%
MA	0.782411185	0.78346891	0.781285119
Ab_Error	0.0029869784	0.001929253	0.004113044
Per_Error	0.3803139%	0.245640121%	0.523689027%

In table-2, the results of example-1 are shown for *n* =13,14 and 15. Similar trend in terms of accuracy has been obtained. The order of accuracy has been improved.

Table-3, Example 2: $\int_1^2 x\sqrt{x+1}\,dx = 2.394157675$

Rules	For n=9	For n=10	For n=11
EM	2.394714891	2.394609023	2.394530692
Ab_Error	0.000557215	0.000451347	0.000373016
Per_Error	0.0232739474%	0.0188520165%	0.01558026%
PMA	2.394213311	2.39415769	2.394188088
Ab_Error	0.000055635	1.499999908e-8	0.000030412
Per_Error	0.0023237817%	0.00000063%	0.00127026%
MA	2.394157718	2.394157674	2.39418808
Ab_Error	4.3000000005e-8	1.00000008e-9	0.000030404
Per_Error	0.0000017960%	0.00000004%	0.00126992%

In table-3, the numerical results are obtained using EM, PMA and MA. We found that the Per_Error for EM methods for n=9, 10, 11 is very large in comparison of other two methods *i.e*. PMA and MA. The Per_Error for *n*=9 using proposed modified algorithm is 10^{-6} where as the EM has 10^{-2} and PMA has 10^{-5}. Thus the proposed method shows improved order of accuracy. Similar trend is observed for *n*=10. However, for *n* = 11, the accuracy of the proposed method almost coincides with PMA at 4th decimal place. Fluctuation has been noticed for larger n, as seen in example-1.

Table-4, Example 3: $\int_0^1 xe^{x^2}\,dx = 0.8591409142$

Rules	For n=13	For n=14	For n=15
EM	0.862664226	0.862179431	0.861788193
Ab_Error	0.003523311	0.003038516	0.002647278
Per_Error	0.41009699%	0.35366911%	0.30813083%
PMA	0.860034834	0.859147486	0.859737179
Ab_Error	0.000893919	0.000006572	0.000596265
Per_Error	0.10404801%	0.00076495%	0.06940247%
MA	0.859167420	0.859141382	0.859733843
Ab_Error	0.000026505	0.000000468	0.000592929
Per_Error	0.00308506%	0.00005447%	0.06901417%

In table-4, the results of Example-3 are given. For *n*=13 and 14, the proposed attains improved order accuracy in comparison of PMA and EM. However, for *n*=15, the accuracy coincides with PMA.

Graphically representation of Table-1,2,3, and 4 are shown in Figures. 1, 2, 3 and 4 respectively. Here,*Y*-axis shows exact and numerical solutions of given integral computed by method of Md.Amanat Ullah (PMA), Proposed modified algorithm (MA) and existing methods (EM) including Trapezoidal rule, Simpson's $\frac{1}{3}$ rule, Simpson's $\frac{3}{8}$ rule and Weddle's rule and number of subintervals are given on *X*-axis. It can be seen here that the proposed method (MA) has the best accuracy in comparison of existing methods (EM). However, the accuracy of the proposed method (MA) is slightly improved in comparison of PMA for higher *n*.

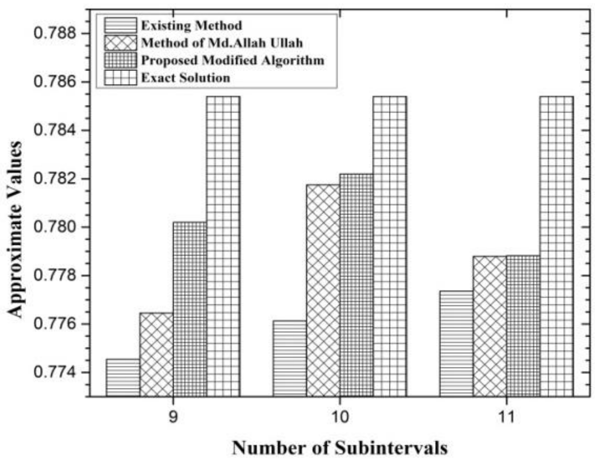

Figure 1: Comparison of integral in Table-1 with Existing method, proposed rule of Md.Amanat Ullah and present Modified algorithm.

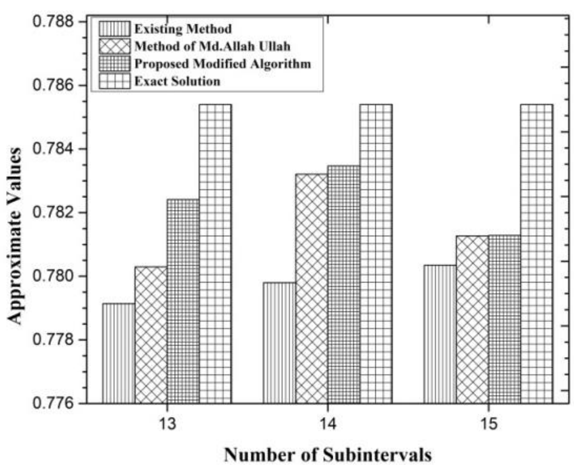

Figure 2: Comparison of integral in Table-2 with Existing method, proposed rule of Md.Amanat Ullah and present Modified algorithm.

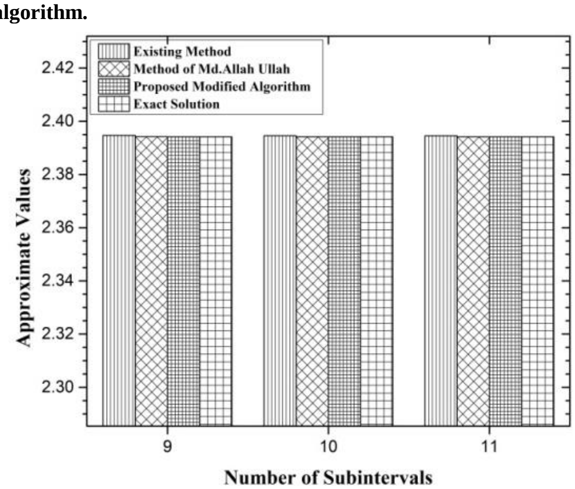

Figure 3: Comparison of integral in Table-3 with Existing method, proposed rule of Md.Amanat Ullahand present Modified algorithm.

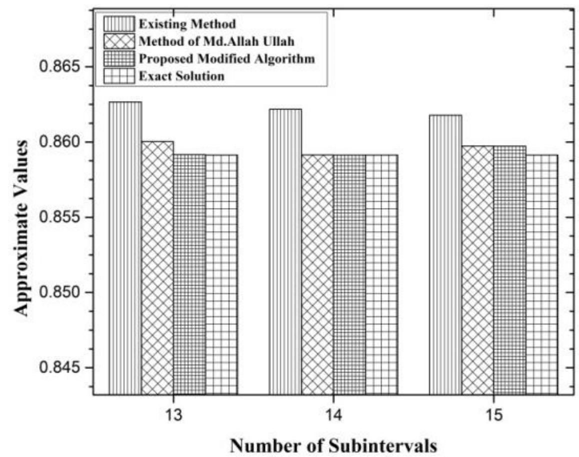

Figure 4: Comparison of integral in Table-4 with Existing method, proposed rule of Md.Amanat Ullah and present Modified algorithm.

4. CONCLUSION

A modified numerical integration technique (algorithm) has been proposed. The proposed technique is a modification of the proposed rule of (Md.Amanat Ullah, 2015). It can be observed that for given number of subintervals the new modified algorithm gives better accuracy when compared with existing rules and proposed rule of Md. Amanat Ullah.It has been noticed that the proposed algorithm produces fluctuation in accuracy when the domain is decomposed into larger sub-intervals(as *n* increases). This fluctuation may be due to the inclusion of trapezoidal rule in the proposed method in combination of other integral schemes. The issue of fluctuation in accuracy will be investigated in future work.

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