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Ratios of the Ring Class Numbers and Class Numbers of a Real Quadratic Field

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Abstract: For a non-square integer $n > 1$, let $K = \boldsymbol{Q}(\sqrt{n})$ be a real quadratic field. In this paper we prove the existence of a new family of infinitely many rings of conductors > 1 whose ratios of ring class numbers and class numbers of K are divisible by a given power of 2 as an extension of our previous work. In addition, we extend the family in our previous work to a countable number of families, each consisting of infinitely many rings of conductors > 1 such that the ratios for each successive family are exactly divisible by a progressively higher power of 2.

Keywords: Real quadratic field, Ring class number, Class number.

1. INTRODUCTION

Let K be a real quadratic field $\boldsymbol{Q}(\sqrt{n})$ over the rationals $\mathbf{Q}$ and d be the field discriminant of K. For an integer $n = df^2, f$ denotes the conductor of the ring $Z[1, f\omega]$, which is a subring of the ring $Z[1,\omega]$ of integers in K over the ring $\boldsymbol{Z}$ of rational integers. Here $h_+(d)$ and $h_+(df^2)$ denote the class number and the ring class number of K in the narrow sense, respectively. In (Tariq *et al.*, 2016) the authors showed that for a canonical decomposition $\prod_{j=1}^{r} f_j$ of f into odd primes f_j such that $f_j = 2s_j - 1$ remained inert in K $(1 \le j \le r)$, the ratio $\frac{h_+(df^2)}{h_+(d)}$ was divisible by the product of powers of distinct primes and that there existed infinitely many such rings exactly divisible by a power of 2. We now extend our result to odd primes of the form $f_j = 2s_j + 1$ that are completely decomposed in K $(1 \le j \le r)$ proving the existence of another family of infinitely many such rings of conductors $f > 1$ whose ratios $\frac{h_+(df^2)}{h_+(d)}$ are divisible by the product of powers of distinct primes and exactly divisible by a power of 2. In addition, we recognize that our previous family can be extended to a countable number of families of infinitely many rings of conductors > 1 whose ratios are exactly divisible by an increasing power of 2 for each subsequent family of the collection.

2. PRELIMINARIES

We state the following two lemmas which are fundamental to this work.

Lemma 2.1 (Tariq *et al.*, 2016) Let K be a real quadratic field with the field discriminant d and ε be the fundamental unit > 1 of K. Then for an odd prime f, it holds that

$$\varepsilon^{f-1} \equiv 1 \pmod f \;\; if \; \left(\frac{d}{f}\right) = 1, \qquad (1)$$

$$\varepsilon^{f+1} \equiv \pm 1 \pmod f \;\; if \; \left(\frac{d}{f}\right) = -1. \qquad (2)$$

Let E be the minimum exponent > 0 such that $\varepsilon \equiv \pm 1 \pmod f$. Then it holds that $E \mid f+1$. Here $\left(\frac{\cdot}{f}\right)$ means the Legendre symbol.

Lemma 2.2 (Alaca and Williams, 2004, Hasse, 1964) Let K be a real quadratic field of prime discriminant $p \equiv 1 \pmod 4$. Then the norm of the fundamental unit is equal to -1.

It is known that $h_+(d) = 2\,h(d)$ if $N_K(\varepsilon) = +1$ and $h_+(d) = h(d)$ if $N_K(\varepsilon) = -1$ or K is an imaginary quadratic field with the fundamental unit ε of K and the field discriminant d. We denote by Z_f the ring $\boldsymbol{Z}[1, f\omega]$ of conductor f with $\omega = \frac{d+\sqrt{d}}{2}$ in the ring $Z_K = \boldsymbol{Z}[1,\omega]$ of integers in K. By the definition of ring class number, $h_+(df^2)$ coincides with the order $\#(A_f/P_f)$ of the factor group A_f/P_f for the fractional ideal group A_f and the principal ideal subgroup P_f of A_f in the ring Z_f under the equivalence relation $\mathfrak{A} \sim \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in A_f$ if there exists $\gamma \in Z_f$ such that $\mathfrak{B} = \gamma\mathfrak{A}$ with $N_K(\gamma) > 0$. Now we state the ring class number formula.

Theorem 2.3 (Cohn, 1994) Let $K = \boldsymbol{Q}(\sqrt{df^2})$ be a quadratic field with the field discriminant d and the conductor f. Then the ring class number formula holds;

$$\begin{cases} h_+(df^2) = h_+(d)f\prod_{p|f}(1 - \frac{\left(\frac{d}{p}\right)}{p})/E_+ \\ h(df^2) = h(d)f\prod_{p|f}(1 - \frac{\left(\frac{d}{p}\right)}{p})/E \end{cases}$$

with the products over the primes $p|f$. Here, if $d < 0$, $h_+(d) = h(d)$and$E_+ = 1$ holds, except $E_+ = 2$ or 3 for $d = -4$ or -3, respectively. If $d > 0, E_+$(resp. E) denotes the exponent of the least power of the totally positive fundamental unitε_+ (resp. fundamental unit ε) such that ${\varepsilon_+}^{E_+}$(resp.ε^E) belongs to the ring $Z_f = \boldsymbol{Z}[1, f\omega]$, where $\omega = \frac{d+\sqrt{d}}{2}$ and $\left(\frac{d}{p}\right)$ denotes the Kronecker symbol.

The contribution of this paper is described in the following section 3.

3. RESULTS

We now extend the main result of (Tariq *et al.*, 2016) to odd primes of the form $f_j = 2q_j + 1$ that are completely decomposed in K $(1 \leq j \leq r)$ proving the existence of another family of infinitely many such rings of conductors $f > 1$ whose ratios $\frac{h_+(df^2)}{h_+(d)}$ are exactly divisible by a power of 2.

Theorem 3.1 Let K be a real quadratic field with the prime discriminant $p \equiv 1 (\mathrm{mod}\ 4)$ and$f$be the conductor $\prod_{j=1}^{r} f_j$ of the ring$\boldsymbol{Z}[1, f\omega]$ with odd prime factorsf_j such that $f_j = 2.q_j + 1$, with odd numbers q_j, $\left(\frac{p}{f_j}\right) = 1$ and $f_j > u_0v_0 (1 \leq j \leq r)$, where $\frac{u_0+v_0\sqrt{p}}{2}$ is the fundamental unit> 1 of K. Put $Z_f = \boldsymbol{Z}[1, f\omega]$. Then there exist infinitely many rings Z_f such that $2^{r-1} \parallel \frac{h_+(pf^2)}{h_+(p)}$.

Proof Since $f_j = 2.q_j + 1$ and $\left(\frac{p}{f_j}\right) = 1$ $(1 \leq j \leq r)$, it follows from Lemma 2.1 that $E_{f_j+}| \ 2.q_j$, where q_j is an odd number $\prod_{i=1}^{s} q_{j_i}$ with odd primes q_{j_i}. From Lemma 2.2,$E_{f_j+}|E_{f_j}$, since ${\varepsilon_+}^{E_{f_j}} = (\varepsilon^2)^{E_{f_j}} = (\varepsilon^{E_{f_j}})^2 \in Z_{f_j}$. Thus $E_{f_j+}|2.q_j$. Hence it is deduced that i) $E_{f_j+} = 1$,
ii) $E_{f_j+} = 2$,iii) $E_{f_j+} = q_{l_1} \cdots q_{l_k}$ or
iv) $E_{f_j+} = 2.q_{l_1} \cdots q_{l_k}for\{l_1, \cdots, l_k\} \subseteq \{j_1, \cdots, j_s\}$.

i) For $\varepsilon = \frac{u_0+v_0\sqrt{p}}{2}$, by Lemma 2.2 we have ${\varepsilon_+}^1 = \varepsilon^2 = \frac{({u_0}^2+{v_0}^2p)/2+u_0v_0\sqrt{p}}{2}$with$f_j \nmid u_0v_0$, so${\varepsilon_+}^1 \notin Z_{f_j}$ $(1\leq j \leq r)$. Thus $E_{f_j+} \neq 1$.

ii) If${\varepsilon_+}^2 \in Z_{f_j}$, then for ${\varepsilon_+}^2 = \frac{u_2+v_2\sqrt{p}}{2}$ with $v_2 = u_1v_1 = \left[\frac{{u_0}^2+{v_0}^2p}{2}\right]u_0v_0$, it follows that $v_2 \equiv 0 (\mathrm{mod} t)$ holds for any $t|f$. For $f_j|f$ it holds that $f_j > u_0v_0$ which gives ${u_0}^2 + {v_0}^2p \equiv 0 (\mathrm{mod}\ f_j)$. Thus it deduces that $\left(\frac{-p}{f_j}\right) = 1$ which is a contradiction to our choice of f_j, where $f_j \equiv 3 (\mathrm{mod}\ 4)$ and $\left(\frac{p}{f_j}\right) = 1$. Therefore, it follows that $E_{f_j+} \neq 2$ $(1 \leq j \leq r)$.

iii) If $E_{f_j+}|\ q_j = q_{j_1} \cdots q_{j_s}$ then from ${\varepsilon_+}^{q_j} = \varepsilon^{2.q_j} = \varepsilon^{f_j-1} \equiv \varepsilon.\varepsilon^{-1} = 1 (\mathrm{mod}\ f_j)$ it follows that ${\varepsilon_+}^{q_j/2} \in Z_{f_j}$ which is a contradiction to the assumption of E_{f_j+} since q_j is odd. Thus $E_{f_j+} \neq q_{l_1} \cdots q_{l_k}$for $\{l_1, \cdots, l_k\} \subseteq \{j_1, \cdots, j_s\}$. Then the case iv) $E_{f_j+} = 2.q_{l_1} \cdots q_{l_k}$ only holds for $\{l_1, \cdots, l_k\} \subseteq \{j_1, \cdots, j_s\}$. Thus by the ring class number formula, for $f = \prod_{j=1}^{r} f_j$, we obtain

$$\frac{h_+(pf^2)}{h_+(p)} = f\prod_{j=1}^{r}\frac{\frac{f_j-1}{f_j}}{E_+} = \frac{\prod_{j=1}^{r}(f_j-1)}{\mathrm{lcm}\,[E_{f_1+}, E_{f_2+}, \cdots, E_{f_r+}]} =$$

$$2^r.\frac{\prod_{j=1}^{r} q_j}{\mathrm{lcm}\,[E_{f_1+}, E_{f_2+}, \cdots, E_{f_r+}]},$$ which deduces that

$2^{r-1} \parallel \frac{h_+(pf^2)}{h_+(p)}$. We show that this family of rings $\boldsymbol{Z}[1, f\omega]$ consists of infinitely many Z_f if each conductor f is a product of odd primes f_j such that $f_j = 2.q_j + 1$ with odd $q_j$and $\left(\frac{p}{f_j}\right) = 1$. For an odd prime f_1 such that $\left(\frac{p}{f_1}\right) = \left(\frac{f_1}{p}\right) = 1 = \left(\frac{n_p}{p}\right)$ with $(f_1, p) = 1$ and a quadratic residue n_p modulo p, it follows that there exist infinitely many primes f_1' congruent to $f_1 (\mathrm{mod}\ p)$ and congruent to $n_p (\mathrm{mod}\ p)$ from Dirichlet's Theorem on Arithmetic Progression. For odd primes f_1' we can choose $f_1' = 2.q_1' + 1$ such that $f_1' \equiv 3 (\mathrm{mod}\ 4)$ with an odd q_1' and $f_1' \equiv n_p (\mathrm{mod}\ p)$. This completes the proof of the existence of infinitely many rings of conductors f whose ratios of the ring class numbers and the class numbers are exactly divisible by a given power of 2.

The following experiments owe to GP/PARI Version 2.7.3. This example is an illustration of Theorem 3.1.

Example 1 Let $K = \boldsymbol{Q}(\sqrt{p})$ with $p = 13$. For $f = \prod_{j=1}^{r} f_j$, let $f_j = 2.q_j + 1$ with odd numbers $q_j$and $\left(\frac{p}{f_j}\right) = 1$ $(1 \leq j \leq r)$. Then we see that $2^{r-1} \parallel \frac{h_+(df^2)}{h_+(d)}$.

j	$f_j = 2.q_j + 1$	$\left(\frac{13}{f_j}\right)$	E_{f_j+}
1	$43 = 2.3.7 + 1$	1	$2.3.7$
2	$79 = 2.3.13 + 1$	1	2.13
3	$103 = 2.3.17 + 1$	1	$2.3.17$
4	$107 = 2.53 + 1$	1	2.53
5	$127 = 2.3^2.7 + 1$	1	$2.3^2.7$
6	$131 = 2.5.13 + 1$	1	2.5

$\frac{h_+(pf_1{}^2)}{h_+(p)} = 1 = \mathbf{2^0}$, $\frac{h_+(p(f_1.f_2)^2)}{h_+(p)} = 6 = \mathbf{2^1}.3$, $\frac{h_+(p(f_1.f_2.f_3)^2)}{h_+(p)} = 36 = \mathbf{2^2}.3^2$,

$\frac{h_+(p(f_1.f_2.f_3.f_4)^2)}{h_+(p)} = 72 = \mathbf{2^3}.3^2$,

$\frac{h_+(p(f_1.f_2.f_3.f_4.f_5)^2)}{h_+(p)} = 3024 = \mathbf{2^4}.3^3.7^1$,

$$\frac{h_+(p(f_1.f_2.f_3.f_4.f_5.f_6)^2)}{h_+(p)} = 78624 = \mathbf{2^5}.3^3.7^1.13^1.$$

The next characterizes the equality of the ratio $\frac{h_+(pf^2)}{h_+(p)}$ to the power of 2.

Theorem 3.2 Let K be a real quadratic field with prime discriminant $p \equiv 1(\text{mod } 4)$. Let$\prod_{j=1}^{r} f_j$be the canonical decomposition of the conductor$f$with odd prime factors $f_j = 2.q_j + 1$ for odd primes$q_j$such that$\left(\frac{p}{f_j}\right) = 1$and $f_j > u_0v_0 (1 \le j \le r)$for the fundamental unit$\frac{u_0+v_0\sqrt{p}}{2} > 1ofK$. Then for the ratio of the ring class number$h_+(pf^2)$and the class number$h_+(p)$of$K$in the narrow sense, it holds that

$$\frac{h_+(df^2)}{h_+(d)} = 2^{r-1}.$$

Proof Since $f_j = 2.q_j + 1$ and $\left(\frac{p}{f_j}\right) = 1\ (1 \le j \le r)$, it follows from Theorem 3.1 that $E_{f_j+} = 2.q_j$ only holds since q_j is prime and $\varepsilon_+{}^1, \varepsilon_+{}^2 \notin Z_{f_j}$ $(1\le j \le r)$ as $f_j > u_0v_0$ .Therefore, by the ring class number formula, for $f = \prod_{j=1}^{r} f_j$, we have $\frac{h_+(pf^2)}{h_+(p)} = f\prod_{j=1}^{r}(\frac{f_j-1}{f_j})/E_+ = 2^r.\frac{q_1.q_2\cdots q_r}{\text{lcm}\,[E_{f_1+},E_{f_2+},\cdots,E_{f_r+}]}$=$2^{r-1}$.

The next example shows several applications of Theorem 3.2.

Example 2 Let $K = \boldsymbol{Q}(\sqrt{p})$with $p = 17$. For $f = \prod_{j=1}^{r} f_j$, let $f_j = 2.q_j + 1$ with distinct odd primes q_j and $\left(\frac{p}{f_j}\right) = 1, (1 \le j \le r)$. that $\frac{h_+(pf^2)}{h_+(p)} = 2^{r-1}$.

j	$f_j = 2.q_j + 1$	$\left(\frac{17}{f_j}\right)$	E_{f_j+}
1	$43 = 2.23 + 1$	1	2.23
2	$59 = 2.29 + 1$	1	2.29
3	$359 = 2.179 + 1$	1	2.179
4	$383 = 2.191 + 1$	1	2.191
5	$467 = 2.233 + 1$	1	2.233

$\frac{h_+(pf_1{}^2)}{h_+(p)} = \mathbf{2^0}$, $\frac{h_+(p(f_1.f_2)^2)}{h_+(p)} = 2 = \mathbf{2^1}$,
$\frac{h_+(p(f_1.f_2.f_3)^2)}{h_+(p)} = 4 = \mathbf{2^2}$, $\frac{h_+(p(f_1.f_2.f_3.f_4)^2)}{h_+(p)} = 8 = \mathbf{2^3}$,
$\frac{h_+(p(f_1.f_2.f_3.f_4.f_5)^2)}{h_+(p)} = 16 = \mathbf{2^4}$.

The association between the canonical decomposition of $f_j \pm 1$ for each odd factor f_j of f and the prime decomposition of E_{f_j+} becomes quite obvious in the next theorem. By taking $f_j = 2^2.q_j - 1$ such that $\left(\frac{p}{f_j}\right) = -1$, where p is the discriminant of K, we show that $2^2|\ E_{f_j+}$.

Theorem 3.3 For a real quadratic field K with a prime discriminant $p \equiv 1(\text{mod } 4)$ and a conductor$f$of the ring $\boldsymbol{Z}[1, f\omega]$ with r odd prime factors f_j such that $f_j = 2^2.q_j - 1$ with odd numbersq_j,$\left(\frac{p}{f_j}\right) = -1$and $f_j > u_1v_1 (1 \le j \le r)$, where$\frac{u_1+v_1\sqrt{p}}{2}$ is the totally positive fundamental unit $\varepsilon_+ > 1$of$K$, there exist infinitely many rings$Z_f = \boldsymbol{Z}[1, f\omega]$ such that $2^{2(r-1)} \parallel \frac{h_+(pf^2)}{h_+(p)}$.

ProofSince $f_j = 2^2.q_j - 1$ and $\left(\frac{p}{f_j}\right) = -1, (1 \le j \le r)$, it follows that $E_{f_j+}|\ 2^2.q_j$, where q_j is an odd number $\prod_{i=1}^{s} q_{j_i}$ with odd primes q_{j_i}. Hence it is deduced that $E_{f_j+} = 1, E_{f_j+} = 2, E_{f_j+} = 2^2$, $E_{f_j+} = q_{l_1}\cdots q_{l_k}$, $E_{f_j+} = 2.q_{l_1}\cdots q_{l_k}$ or $E_{f_j+} = 2^2.q_{l_1}\cdots q_{l_k}$ for $\{l_1, \cdots, l_k\} \subseteq \{j_1, \cdots, j_s\}$.

For $\varepsilon = \frac{u_0+v_0\sqrt{p}}{2}$, by Lemma 2.2 we have

$\varepsilon_+{}^1 = \varepsilon^2 = \frac{(u_0{}^2+v_0{}^2p)/2+u_0v_0\sqrt{p}}{2}$with$f_j \nmid u_0v_0$ since $u_1v_1 = \left[\frac{u_0{}^2+v_0{}^2p}{2}\right]u_0v_0 > u_0v_0$, so $\varepsilon_+{}^1 \notin Z_{f_j}$ $(1\le j \le r)$. Thus $E_{f_j+} \neq 1$.

For $\varepsilon_+{}^2 = \left(\frac{u_1+v_1\sqrt{p}}{2}\right)^2 = \frac{(u_1{}^2+v_1{}^2p)/2+u_1v_1\sqrt{p}}{2}$ with $f_j \nmid u_1v_1$, so $\varepsilon_+{}^2 \notin Z_{f_j}$ $(1\le j \le r)$.Thus $E_{f_j+} \neq 2$.

If$\varepsilon_+{}^{2^2} \in Z_{f_j}$, then for $\varepsilon_+{}^{2^2} = \frac{u_4+v_4\sqrt{p}}{2}$ with $v_4 = u_2v_2 = \left[\frac{u_1{}^2+v_1{}^2p}{2}\right]u_1v_1$, it follows that $v_4 \equiv 0(\text{mod}t)$ holds for any $t|f$. For $f_j|f$ it holds that $f_j > u_1v_1$ which gives $u_1{}^2 + v_1{}^2p \equiv 0$ (mod f_j). Since $p \equiv 1(\text{mod } 4)$, we have $u_1{}^2 - v_1{}^2p = 4$. Substituting $v_1{}^2p = u_1{}^2 - 4$in$u_1{}^2 + v_1{}^2p \equiv 0$ (mod f_j) gives $u_1{}^2 \equiv 2$ (mod f_j) from which it follows that $\left(\frac{2}{f_j}\right) = 1$, a contradiction to the assumption of f_j since $\left(\frac{2}{f_j}\right) = -1$ for $f_j \equiv 3(\text{mod } 8)$. On the other hand, by substituting $u_1{}^2 = v_1{}^2p + 4$ in $u_1{}^2 + v_1{}^2p \equiv 0$ (mod f_j), we get $(v_1p)^2 \equiv -2p$ (mod f_j) implying that $\left(\frac{-2p}{f_j}\right) = 1$. But

$\left(\frac{-2p}{f_j}\right) = \left(\frac{-1}{f_j}\right)\left(\frac{2}{f_j}\right)\left(\frac{p}{f_j}\right) = (-1)(-1)(-1) = -1$ for $f_j \equiv 3(\text{mod } 8)$giving$f_j \equiv 3(\text{mod } 4)$ which results in a contradiction. Therefore, it follows that $E_{f_j+} \neq 2^2 (1 \leq j \leq r)$. By $\varepsilon_+{}^{2.q_j} = \varepsilon_+{}^{\frac{f_j+1}{2}} = \varepsilon^{f_j+1} \equiv \varepsilon^{\sigma}.\varepsilon = -1(\text{mod } f_j)$, since $p \equiv 1(\text{mod } 4)$, it follows that $(\varepsilon_+{}^{q_j})^2 \equiv -1\ (\text{mod} f_j)$ but $\left(\frac{-1}{f_j}\right) = 1$ is a contradiction to the assumption of f_j. Thus $E_{f_j+} \neq 2.q_{l_1} \cdots q_{l_k}$for $\{l_1, \cdots, l_k\} \subseteq \{j_1, \cdots, j_s\}$. This also rules out the possibility $E_{f_j+} = q_{l_1} \cdots q_{l_k}$. Then the case $E_{f_j+} = 2^2.q_{l_1} \cdots q_{l_k}$only holds for $\{l_1, \cdots, l_k\} \subseteq \{j_1, \cdots, j_s\}$. Thus by the ring class number formula, for $f = \prod_{j=1}^{r} f_j$, we obtain

$\frac{h_+(pf^2)}{h_+(p)} = f \prod_{j=1}^{r} \frac{\frac{f_j+1}{f_j}}{E_+} = \frac{\prod_{j=1}^{r}(f_j+1)}{\text{lcm}\,[E_{f_1+}, E_{f_2+}, \cdots, E_{f_r+}]} = 2^{2r}.\frac{\prod_{j=1}^{r} q_j}{\text{lcm}\,[E_{f_1+}, E_{f_2+}, \cdots, E_{f_r+}]}$, which deduces that $2^{2(r-1)} \parallel \frac{h_+(pf^2)}{h_+(p)}$. We show that the family of rings $\boldsymbol{Z}[1, f\omega]$ consists of infinitely many Z_f if each conductor f is a product of odd primes f_j such that $f_j = 2^2.q_j - 1$ with odd $q_j$and $\left(\frac{p}{f_j}\right) = -1$. For an odd prime f_1 such that $\left(\frac{p}{f_1}\right) = \left(\frac{f_1}{p}\right) = -1 = \left(\frac{n_p}{p}\right)$ with $(f_1, p) = 1$ and a quadratic non-residue n_p modulo p, it follows that there exist infinitely many primes $f_1{}'$ congruent to $f_1\,(\text{mod } p)$ and congruent to $n_p\,(\text{mod } p)$ from Dirichlet's Theorem on Arithmetic Progression. For odd primes $f_1{}'$ we can choose $f_1{}' = 2^2.q_1{}' - 1$ such that $f_1{}' \equiv 3(\text{mod } 8)$ with an odd $q_1{}'$ and $f_1{}' \equiv n_p(\text{mod } p)$. This completes the proof of the existence of infinitely many rings of conductors f whose ratios of the ring class numbers and class numbers are exactly divisible by a much higher power of 2 as compared to Theorem 4.2 of (Tariq *et al.*, 2016).

By stating the next theorem, we have proved that there exist a countable number of families of infinitely many rings Z_f whose ratios of the ring class numbers and class numbers are exactly divisible by an increasing power of 2 with each successive family in the set. This countable collection is a consequence of Theorem 4.2 (Tariq *et al.* 2016), Theorem 3.3 and the next theorem weighed together. The proof of Theorem 3.4 trails an outline of the proof of Theorem 3.3 with modifications for $f_j \equiv 7(\text{mod } 8)$ $f_j \equiv 3(\text{mod } 8)$ in Theorem 3.3.

Theorem 3.4 Let$K$be a real quadratic field with the prime discriminant$p \equiv 1(\text{mod } 4)$and f be the conductor$\prod_{j=1}^{r} f_j$of the ring$\boldsymbol{Z}[1, f\omega]$with odd prime factors$f_j$such that$f_j = 2^n.q_j - 1$with odd numbers$q_j$, $n > 2$,$\left(\frac{p}{f_j}\right) = -1andf_j > u_0 v_0 (1 \leq j \leq r)$, $\frac{u_0 + v_0\sqrt{p}}{2}$ is the fundamental unit$>1$of$K$. Then there exist infinitely many rings$Z_f = \boldsymbol{Z}[1, f\omega]$such that$2^{n(r-1)} \parallel \frac{h_+(pf^2)}{h_+(p)}$.

The next example affirms the efficacy of Theorem 3.4.

Example 3 Let $K = \boldsymbol{Q}(\sqrt{p})$with $p = 17$. For $f = \prod_{j=1}^{r} f_j$,$n = 3$, let $f_j = 2^n.q_j - 1$ with odd numbers q_j and $\left(\frac{p}{f_j}\right) = -1\ (1 \leq j \leq r)$. Then we see that $2^{n(r-1)} \parallel \frac{h_+(pf^2)}{h_+(p)}$.

j	$f_j = 2^n.q_j - 1$	$\left(\frac{17}{f_j}\right)$	E_{f_j+}
1	$23 = 2^3.3 - 1$	-1	$2^3.3$
2	$71 = 2^3.3^2 - 1$	-1	$2^3.3^2$
3	$167 = 2^3.3.7 - 1$	-1	$2^3.7$
4	$199 = 2^3.5^2 - 1$	-1	$2^3.5^2$
5	$439 = 2^3.5.11 - 1$	-1	$2^3.5.11$

$\frac{h_+(pf_1{}^2)}{h_+(p)} = 1 = 2^0 = \boldsymbol{2^{3(1-1)}}$,

$\frac{h_+(p(f_1.f_2)^2)}{h_+(p)} = 24 = 2^3.3^1 = \boldsymbol{2^{3(2-1)}}.3^1$,

$\frac{h_+(p(f_1.f_2.f_3)^2)}{h_+(p)} = 576 = 2^6.3^2 = \boldsymbol{2^{3(3-1)}}.3^2$,

$\frac{h_+(p(f_1.f_2.f_3.f_4)^2)}{h_+(p)} = 4608 = 2^9.3^2 = \boldsymbol{2^{3(4-1)}}.3^2$,

$\frac{h_+(p(f_1.f_2.f_3.f_4.f_5)^2)}{h_+(p)} = 184320 = 2^{12}.3^2.5^1 = \boldsymbol{2^{3(5-1)}}.3^2.5^1$.

4. CONCLUSION

A comprehensive relationship between the prime decompositions of $f_j \pm 1$ and E_{f_j+} is observed which needs a further study into the phenomena of parallel decompositions of $f_j \pm 1$and E_{f_j+}. This would allow to investigate whether there exists infinitely many rings of conductors > 1 whose ratios are exactly divisible by a power of an oddprime p.

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