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A new Simpson's 1/3-type quadrature scheme with geometric mean derivative for the Riemann-Stieltjes integral

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Abstract

The main purpose of this research is to develop and improve the Simpson's 1/3-type quadrature scheme numerically utilizing the geometric mean derivative for the Riemann-Stieltjes integral. The proposed scheme of Simpson's 1/3-type is described in basic form and also in composite form. The performance of the proposed scheme is compared with existing schemes by experimental results using MATLAB. It has been noted in numerical results that the performance of new proposed scheme is more efficient against the existing schemes in terms of errors, computational cost, and average CPU time.

Keywords: Geometric Mean, Simpson's 1/3 rule, Riemann-Stieltjes integral, Computational cost, Time efficiency

Introduction

Numerical integration has been used to estimate a numeric value of a definite integral and it has several applications in engineering where the area of the function is computed by the curve. The definite integrals of such functions $f(x) = e^{x^2}$ and $f(x) = \sin x^2$ cannot be evaluated analytically, so that they are solved numerically. The numerical method for the evaluation of definite integral is known as quadrature. Most of the work has been done on the numerical integration for the Riemann integral in the literature. However, little work has been focused on numerical integration for the Riemann-Stieltjes integral.
Riemann-Stieltjes integral (RS-integral) is defined in [1] as

$\int_a^b f(x)\, d\alpha(x)$, where f is integrand and α is integrator.

RS-integrals have several applications in the fields of Operator theory, Functional analysis, Complex analysis, Statistics and probability theory and others.

In literature, the following papers [2] and [3] presented the derivative-based closed Newton-Cotes quadrature schemes for the Riemann integral. However, [4] presented a new four-point closed quadrature rule by the modification of [2] in Simpson's 3/8 rule using midpoint derivative. Shaikh, [5] discussed the numerical solution of integral equations using quadrature method. In literature, the following papers [6] and [7] presented the inequalities on quadrature rules for the RS-integral. Zhao *et al.*, [8] presented the quadrature rule of trapezoid-type for the RS-integral using the midpoint derivative.

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Zhao *et al.*, [9] proposed the composite form of trapezoid rule for the RS-integral using derivative-based approach. Memon *et al.,* [10] modified the [8] scheme for the RS-integral using experimental work. Memon *et al.,* [11] proposed a new heronian mean derivative-based Simpson's 1/3 scheme for the RS-integral with experimental work.

Definition of quadrature rules for the Riemann integral

The basic formula of a definite integral over the closed interval $[a, b]$ is defined in [1] as

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \qquad (1)$$

In this study, a new derivative-based Simpson's 1/3 scheme is developed for the RS-integral using the geometric mean. The proposed scheme is verified and compared by numerical experiments in terms of errors, cost efficiency and time efficiency.

Materials and Methods

Here n+1 are separate integration points at x_0, x_1, …,x_n inside the closed interval $[a, b]$ and n+1 are weights w_i, i = 0, 1, 2, ..., n. If the points of integration are equally divided over the closed interval $[a, b]$ then

$x = a+ih$, where h = (b-a)/n.

Some Existing Scheme for the RS-integral

The basic forms of some existing schemes: T [6], ZT [8], MZT [10], are described for the RS-integral in (2)-(4) as:

$$T \approx \left(\frac{[illegible]}{[illegible]}\int [illegible] - [illegible]\right)[illegible] + \int [illegible] \qquad (2)$$

$$[illegible] \approx \left(\frac{1}{[illegible]}\int [illegible] - [illegible]\right)[illegible] + \int [illegible] + \left(\int\int^{t} [illegible] - \int [illegible]\right)'' \frac{1}{[illegible]} h [illegible], \qquad (3)$$

$$[illegible]_{[illegible]T} = \frac{(-2[illegible]^2 + [illegible]^2 - [illegible])\int [illegible] + 6[illegible]\int\int^{t} [illegible] - 6\int\int^{t}\int [illegible]}{6\int\int^{t} [illegible] - 3([illegible] - [illegible])\int [illegible]}$$

.

$$[illegible]T \approx \left(\frac{1}{[illegible]}\int [illegible] - [illegible]\right)[illegible] + \int [illegible] + \left(\int\int^{t} [illegible] - \int [illegible]\right)'' \frac{1}{[illegible]} h [illegible], \qquad (4)$$

$$[illegible]_{M[illegible]T} = \frac{(-2[illegible]^2 + [illegible]^2 + [illegible])\int [illegible] + 6[illegible]\int\int^{t} [illegible] - 6\int\int^{t}\int [illegible]}{[illegible]}$$

[illegible]

The composite forms of the CT, ZCT and MZCT schemes are described in (5)-(7) as:

$CT \approx$ [illegible] (5)

$ZCT \approx$ [illegible] (6)

$MZCT \approx$ [illegible] (7)

Where,

$$
E_{\cdots,k} = \frac{(-\cdots + \cdots_{k-\cdots} - \cdots_{k-\cdots}\cdots)\int_{\cdots_{k-\cdots}}^{\cdots_k} \cdots(t)\cdots t + 6\cdots\int_{\cdots}^{\cdots}\int_{\cdots_{k-\cdots}}^{t}\cdots(\cdots)\cdots - 6\int_{\cdots}\int_{\cdots_{k-\cdots}}\int_{\cdots_{k-\cdots}}\cdots(\cdots)\cdots t}{6\int_{\cdots_{k-\cdots}}^{k}\int_{\cdots}^{t}\cdots(\cdots)\cdots - \frac{\cdots(k\ \cdots-\cdots)}{n}\int_{\cdots_{k-\cdots}}^{k}\cdots(t)\cdots t}
$$

$$
E_{M\cdots,k} = \frac{(-\cdots_k + \cdots_{\cdots-\cdots} + \cdots_{\cdots-\cdots}\cdots_k)\int_{\cdots-\cdots}^{\cdots_k}\cdots(\cdots)\cdots + 6\cdots\int_{\cdots-\cdots}^{\cdots_k}\int_{\cdots-\cdots}^{t}\cdots(\cdots)\cdots t - 6\int_{\cdots-\cdots}\int_{\cdots-\cdots}\int_{\cdots-\cdots}\cdots(\cdots)\cdots t}{6\int_{\cdots-\cdots}^{\cdots_k}\int_{\cdots-\cdots}^{t}\cdots(\cdots)\cdots t - \frac{\cdots(\cdots_k - \cdots_{\cdots-\cdots})}{n}\int_{\cdots-\cdots}^{\cdots_k}\cdots(\cdots)\cdots t}
$$

Proposed Geometric Mean Derivative-Based Simpson's 1/3 Scheme for the Riemann-Stieltjes Integral

The basic proposed geometric mean derivative-based Simpson's 1/3 GMS13 scheme for the RS-integral is defined in (8) as

$$
\begin{aligned}
\int_a^b f(t)\,dg \approx GMS13 = {} & \left(\frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt - \frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) \\
& + \left(\frac{4}{b-a}\int_a^b g(t)dt - \frac{8}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f\left(\frac{a+b}{2}\right) \\
& + \left(g(b) - \frac{3}{b-a}\int_a^b g(t)dt + \frac{4}{(b-a)^2}\int_a^b\int_a^t g(x)dxdt\right) f(b) \\
& + \left(\begin{array}{l} \frac{-(b-a)^2(3a+5b)}{96}\int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdydt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \end{array}\right) f^{(4)}\left(\sqrt{ab}\right), \qquad (8)
\end{aligned}
$$

The precision of this scheme is 4.

The composite form of the proposed GMS13 scheme is GMCS13 scheme for the RS-integral is defined in (9) as

$$
\begin{aligned}
\int_a^b f(t)\,dg \approx GMCS13 = {} & \left[\frac{4n^2}{(b-a)^2}\int_a^{x_1}\int_a^t g(x)dxdt - \frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) \\
& + \frac{4n}{b-a}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt - \frac{2n}{b-a}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right] f\left(\frac{x_{k-1}+x_k}{2}\right) \\
& + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\frac{4n}{b-a}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt + \int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right) - \left(3\int_{x_{k-1}}^{x_k} g(t)dt + \int_{x_k}^{x_{k+1}} g(t)dt\right)\right] f(x_k) \\
& \left[-h^2(3x \quad + 5x\ )\int^{x_k} g(t)dt + \frac{17x_k^{\ 2} - 10x_{k-1}x_k - 7x_{k-1}^{\ 2}}{\cdots}\int^{x_k}\int^{t} \right.
\end{aligned}
$$

$$+\sum_{k=1}^{n}\left[\begin{array}{l} \frac{g(x_{k-1})}{96} \int_{x_{k-1}}^{x_k} dx \; dt \quad 48 \int_{x_{k-1}} \int_{x_{k-1}} \\ -x_k \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)\,dxdydt + \int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)\,dxdydzdt \end{array}\right] f^{(4)}\left(x_{k-1}, x_k\right)$$

$$+\left[g(b) - \frac{3n}{b-a}\int_{x_{n-1}}^{b} g(t)\,dt + \frac{4n^2}{(b-a)^2}\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t} g(x)\,dxdt \right] f(b) \qquad (9)$$

Results and Discussion

The performance of proposed GMCS13 scheme for the RS-integral is tested by experimental results in the comparison of existing schemes CT, ZCT and MZCT schemes. Three numerical problems have been tested for each scheme taken from [11], which were determined utilizing MATLAB. The results of all schemes are noted in Intel (R) Core (TM) Laptop having RAM 8.00GB with processing speed 1.00GHz-1.61GHz. Double precision arithmetic is used for numerical results.

Example 1.

$\int_{3.5}^{4.5}$ �i� 5 ��(��� �)=0.227676016130689

Example 2.

$\int_{5}^{6}$ �i� � �(�^{3}) = -59.655908136641912

Example 3.

$\int_{5}^{6}$ �$^{�}$� �i� � = 187.4269314248657

In Figs. 1-3, the absolute errors of proposed GMCS13 scheme have been compared against the existing schemes CT, ZCT and MZCT and finally, it is noted from numerical results that the errors of proposed GMCS13 scheme reduced rapidly whereas the errors of existing schemes reduced slowly for all examples.

Table 1: Computational cost obtained to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

Quadrature Variants	Computational cost		
	Example 1	Example 2	Example 3
CT	439	1415	2503
ZCT	1043	3503	6253
MZCT	73	78	78
GMCS13	59	45	66

Table 2: Average CPU time obtained to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

Quadrature Variants	CPU time (in seconds)		
	Example 1	Example 2	Example 3
CT	68.04	12.82	432.30
ZCT	552.60	153.98	6469.38
MZCT	31.90	6.14	32.54
GMCS13	27.61	5.84	31.20

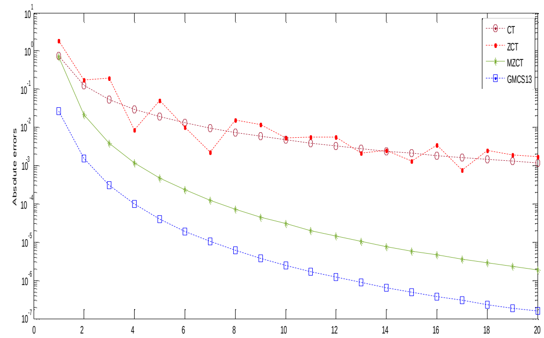

Fig 1. Comparison of absolute errors versus number of rstrips to all schemes for Example 1

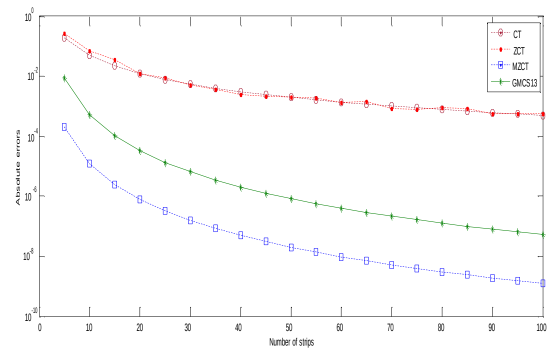

Fig 2. Comparison of absolute errors versus number of strips to all schemes for Example 2

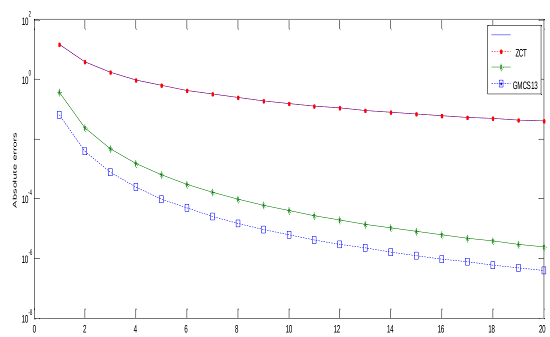

Fig 3. Comparison of absolute errors versus number of strips to all schemes for Example 3

It is observed from Table 1 and Table 2 that the proposed Scheme obtained minimum computational cost and took smaller average CPU time to achieve the error 10^{-5} in comparison of others existing schemes for Examples 1-3.

Conclusion

A new efficient Simpson's 1/3-type quadrature scheme with geometric mean derivative was proposed for the RS-integral. Three numerical problems were examined in order to show the performance of proposed scheme in comparison of three existing schemes. The overall performance of proposed scheme was efficient numerically against the existing schemes.

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