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On the Fundamental Units and a Lower Estimate of the Class Numbers of Certain Real Quadratic Fields

R. QURESHI, T. NAKAHARA[++], B. AHMAD*, S. I. A. SHAH*

National University of Computer & Engineering Science, Peshawar Campus, Khyber Pakhtunkhwa, Pakistan.

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Abstract: We give explicitly the fundamental units of real quadratic fields over the rationals Q, which is distinct from RD type. H. Yokoi and the second author have found an explicit expression of the fundamental units of certain real quadratic fields *K*, which intersect trivially to RD type, an Extended RD and a Generalized RD one. Using the formula of the fundamental units, we give a lower estimate of the class numbers of these fields and determine the positive density of such fields *K*.

Keywords: Real Quadratic Field, Fundamental Unit, Class Number.

1. INTRODUCTION

Let K and ε_D be real quadratic fields $Q(\sqrt{D})$ $(D>0, \text{square free})$ and the normalized fundamental unit of K, respectively. Degert gave explicitly the fundamental unit of real quadratic field $Q(\sqrt{D})$ for $D=n^2+r$ with $-n<r\leqq n$ and $r\,|\,4n$, which is called an RD type (Degert, 1958). He proposed that what will be the fundamental unit if we remove the condition; $-n<r\leqq n$ and if D satisfies $(n\pm m)^2+r_{\pm m}$ for $m\geqq 1$ such that $r_{\pm m}\,|\,4(n\pm m)$ and $r_{\pm 0}\nmid 4n$. We define these fields as a Generalized RD type in Section 2. H. K. Kim have found the necessary and sufficient conditions for the class number one of real quadratic fields of RD type in terms of quadratic polynomials producing prime numbers based on the results of (Byeon and Kim, 1996, Lang, 1968, Siegel, 1969, Yamamura, 2013).

Later on, Mollin removed the condition $-n<r\leqq n$ for RD type and named it an Extended RD type, which is abbreviated by ERD type (Mollin, 1995).

Definition 1.1 *Let K be a real quadratic field* $Q(\sqrt{D})$ *and D be a square free integer which can be written in the from* $D=n^2+r$ *with* $|r|>n$ *and* $r\,|\,4n$ *.Then the field K is said to be of ERD type.*

However, Degert proposed what is the fundamental unit if we remove the condition; $-n<r\leqq n$ and if D satisfies $(n\pm m)^2+r_{\pm m}$ for $m\geqq 1$ such that $r_{\pm m}\,|\,4(n\pm m)$ and $r_{\pm 0}\nmid 4n$ with $r_{\pm 0}=r$. We may define such fields as a Generalized RD type, which is abbreviated by GRD type (Degert, 1958).

Definition 1.2 *LetK be a real quadratic field* $Q(\sqrt{D})$ *and* D *be a non-square, namely* $D=f^2d$ *with the field discriminant d, which can be written in the form*:

$D=(n-m)^2+r_{-m}$ *and* $r_{-m}\,|\,4(n-m)$ *with* $m\geqq 0$ *or,*

$D=(n+m)^2+r_{+m}$ *and* $r_{+m}\,|\,4(n+m)$ *with* $m\geqq 0$, *where* $r_{\pm m}\neq \pm 1, \pm 4$. *Then the field K is said to be of GRD type.*

Thus from the above three definitions we can recognize that if the conductor f of the ring $Z[1, fw]$ with $w=\dfrac{d+\sqrt{d}}{2}$ is bigger than 1 in the definition of GRD, then three families F_{RD} of RD type, F_{ERG} of ERD type and F_{GRD} of GRD type are linearly disjoint to each other, that is, for any fields $K\in F_{RD}, L\in F_{ERD}$ and $M\in F_{GRD}$, it holds that if $f>1$, then K, L, M intersect trivially to each other;

$$K\cap M=L\cap M=Q,$$

and if $f=1$, then $K\cap L=Q$ and for the case (iii) in lemma 1.3 and $m=0$ in Definition 1.2, $K=M$. The

[++]Corresponding author, Email: inayat64@gmail.com,

*Department of Mathematics, Islamia College Peshawar, Khyber Pakhtunkhwa, Pakistan.

rabiaqureshi1981@yahoo.com, toru.nakahara@nu.edu.pk, nakahara@ms.saga-u.ac.jp

fundamental units of RD type are determined by the following lemma (Degert, 1958).

Lemma 1.3 *Let $K = Q(\sqrt{D})$ be a real quadratic field with a square free integer $D > 1$. Let $\in_D$ be the fundamental unit of K and put $D = n^2 + r$ with $n, r \in Z, -n < r \leqq n$. If $4n \equiv 0 (\mathrm{mod}\ r)$ holds, then ε_D is of the following forms:*

(i) $\varepsilon_D = n + \sqrt{D}$ with $N(\varepsilon_D) = -\operatorname{sgn} r$ *for* $|r| = 1$ (except for $D = 5$),

(ii) $\varepsilon_D = \dfrac{n + \sqrt{D}}{2}$ with $N(\varepsilon_D) = -\operatorname{sgn} r$ *for* $|r| = 4$, *and,*

(iii) $\varepsilon_D = \dfrac{2n^2 + r + 2n\sqrt{D}}{|r|}$ with $N(\varepsilon_D) = 1$ *for* $|r| \neq 1, 4$.

Here $N(\xi)$ denotes the norms $\xi\xi^{\sigma}$ of a number ξ of K with respect to K/Q for the embedding σ of K being not the identity.

2. FUNDAMENTAL UNITS OF A GRD TYPE OF REAL QUADRATIC FIELDS

With slight modifications of Lemma 1.3, the fundamental unit of the GRD type is exhibited in the following theorem.

Theorem 2.1 *Let $K = Q(\sqrt{D})$ be of a GRD type with $D = (n \pm m)^2 + r_{\pm m} = f^2 d$ and $m \geqq 1$, where d is the field discriminant. Then the fundamental unit ε_D of the ring $Z_f = Z[1, fw]$ with $w = \dfrac{d + \sqrt{d}}{2}$ is given by*

$$\varepsilon_D = \frac{2(n+m)^2 + r_{+m} + 2(n+m) f\sqrt{d}}{|r_{+m}|} \text{ with}$$

$r_{+m} | 4(n+m)$, and $N(\varepsilon_D) = 1$,

$$\varepsilon_D = \frac{2(n-m)^2 + r_{-m} + 2(n-m) f\sqrt{d}}{|r_{-m}|} \text{ with}$$

$r_{-m} | 4(n-m)$, and $N(\varepsilon_D) = 1$.

Here $r_{+m} = r_0 - (2nm + m^2)$ and $r_{-m} = r_0 + 2nm - m^2$.

Proof: Let $D = (n \pm m)^2 + r_{\pm m}$ with $r_{\pm m} = 2nm - m^2 \neq r_0$, $-n < r_0 \leqq n$ and $r_{\pm m} | 4(n \pm m)$. By the analogue of Lemma 1.3 (iii), we have a unit:

$$\varepsilon_D = \frac{2(n \pm m)^2 + r_{\pm m} + 2(n \pm m) f\sqrt{d}}{|r_{\pm m}|}$$

For the case $r_{+m} = r_0 - 2mn - m^2 < 0$, we claim that

$$\varepsilon_D = \frac{2(n+m)^2 + r_{+m} + 2(n+m) f\sqrt{d}}{|r_{+m}|} > 1$$

is the fundamental unit of Z_f. If ε_D is not the fundamental unit, then there exists a fundamental unit $\varepsilon_f = \dfrac{s + tf\sqrt{d}}{2} > 1$ of Z_f such that $\varepsilon_D \geqq \varepsilon_f{}^2$. Since $s^2 = 4 + t^2 D$, $4(n+m)^2 + 2r_{+m} \geqq \dfrac{1}{2}(4 + t^2 D + t^2 D)|r_{+m}|$ holds. We deduce that $4(n+m)^2 + 2 \cdot r_{+m} \geqq (2 + t^2 D)|r_{+m}| = 2|r_{+m}| + t^2 D|r_{+m}|$, which implies that $4(n+m)^2 + 2 \cdot r_{+m} = 4(n+m)^2 + 4 \cdot r_{+m} - 2 \cdot r_{+m} = 4D - 2 \cdot r_{+m} = 4D + 2(-r_{+m})$ $= 4D + 2|r_{+m}| \geqq 2|r_{+m}| + t^2 D|r_{+m}|$, it holds that $4D \geqq t^2 D|r_{+m}|$. We claim that $t \leqq 3$.

Because for $t = 1$, we have RD type (ii) $s^2 - D = 4$ and for $t = 2$, $s^2 - t^2 D = 4$ implies $4D = (2s')^2 - 4$ with $s = 2s'$, and hence we have again RD type (i). But for $t \geqq 3$, $4D \geqq t^2 D|r + m|$ is impossible. Thus ε_D is the fundamental unit of Z_f.

For the case $r_{-m} = r + 2mn - m^2 > 0$, if ε_D is not a fundamental unit of Z_f, then there exists the fundamental unit $\varepsilon_f = \dfrac{s + t\sqrt{D}}{2} > 1$ such that $\varepsilon_D \geqq \varepsilon_f^2$. By $\varepsilon_D \geqq \varepsilon_f^2$ and with $s^2 = 4 + t^2 D, 4(n-m)^2 + 2r_{-m} \geqq \dfrac{1}{2}(4 + t^2 D + t^2 D)|r_{-m}|$ holds. Thus we deduce that $4(n-m)^2 + 2r_{-m} \geqq (2 - t^2 D) r_{-m}$.

Then we have $4(n-m)^2 + 2r_{-m} \geqq 2r_{-m} + t^2 (n-m)^2 r_{-m} + t^2 r_{-m}{}^2$.

This implies that

$4(n-m)^2 \geqq t^2(n-m)^2 r_{-m} + t^2 r_{-m}{}^2$. But $t \geqq 3$, the inequality becomes a contradiction. Thus ε_D is the fundamental unit of Z_f.

Remark 2.2 The minimum odd field discriminant of a GRD type is 69 = $(8+1)^2 - 12$ for the case of r_{+1} and $\varepsilon_{69} = \frac{2(8+1)^2 - 12 + 2(8+1)\sqrt{69}}{|-12|} = \frac{25+3\sqrt{69}}{2}$. It is also an ERD type. The even one is shown in Remark 2.5.

In a GRD type of

$D = (2n)^2 + r = (2n-1)^2 + r_{-1}$ or $(2n+1)^2 + r_{+1}$

with $n \geqq 3$, the number D is always a composite number by the following proposition [See Example 3.6].

Proposition 2.3 If $Q(\sqrt{D})$ is of GRD type of $D = (2n)^2 + r = (2n-1)^2 + r_{-1}$ or of $(2n+1)^2 + r_{+1}$ with $n \geqq 3$, $-2n < r < 2n$ such that $r \nmid 4(2n)$ and $r_{-1} \mid 4(2n-1)$ or $r_{+1} \mid 4(2n+1)$, then D is not a prime number

Proof: Let $Q(\sqrt{D})$ be a GRD type for $D = (2n)^2 + r = (2n-1)^2 + (r+4n-1)$ with $-2n < r \leqq 2n$, $n \geqq 3$ and $r_{-1} = r + 4n - 1$ such that $r \nmid 4(2n)$ and $5 \leqq 2n-1 < r_{-1} = 4n-1+r \leqq 6n-1$. From $r_{-1} \mid 4(2n-1)$, if $r_{-1} = 2^e$, then $r_{-1} = 1, 2$ and 4 holds, each of which is impossible by $5 < r - 1$. Then these exists the smallest odd prime $p \mid r_{-1}$ and $p \mid 2n-1$. Thus from $p < ps^2 + \frac{r-1}{p}$ for some s, D is not a prime number by $D = p\left(\frac{(2n-1)^2}{p} + \frac{r-1}{p}\right)$.

Let $Q(\sqrt{D})$ be a GRD type for $D = (2n)^2 + r$ with $r \nmid 4(2n)$ such that $D = (2n+1)^2 + r_{+1}$ with $r_{+1} \mid 4(2n+1)$ an $-(6n+1) < r_{+1} = r - (4n+1) \leqq -(2n+1)$. Thus there exists the smallest odd prime $q \mid r_{+1}$ and $q \mid 2n+1$. Then by $D = q\left(\frac{(2n+1)^2}{q} + \frac{r_+}{q}\right) > q$, D is not a prime number. The next theorem is obtained under the consideration of explicit fundamental units of the ring $Z_f = [1, fw]$ of conductor $f \geqq 1$ for a GRD type of real quadratic fields $Q(\sqrt{D})$ where $D = f^2 d$ with the field discriminant d.

Using Theorem 2.1, we have found that for $D = 765 = 3^2 \cdot 85$ with a square part the fundamental unit of $Q(\sqrt{85})$ is the square root of the unit ε_{765} of $Q(\sqrt{765})$. So we extend this phenomenon to a family of real quadratic fields $Q(\sqrt{D})$ with $D = f^2 d$, $d = f^2 a^2 + 4$, where f is a conductor $\geqq 1$ of Z_f.

Theorem 2.4 *Let K be a real quadratic field $Q(\sqrt{D})$ with $D = f^2 d$ and the field discriminant $d = f^2 a^2 + 4$. Then a number $\varepsilon_D = \frac{f^2 a^2 + 2 + af\sqrt{d}}{2}$ is a unit in K. Let ε_0 be the fundamental unit > 1 of $Q(\sqrt{D})$. Then it holds that $\varepsilon_0 = \sqrt{\varepsilon_D}$.*

Proof: Let $D = (f^2 a)^2 + 4f^2 = f^2 d$ and d be the field discriminants of $Q(\sqrt{D})$. Then it follows that for $D = f^2(f^2 a^2 + 4)$ we have

$$\varepsilon_D = \frac{2(f^2 a)^2 + 4f^2 + 2f^2 a\sqrt{f^2(f^2 a^2 + 4)}}{4f^2}$$

$$= \frac{f^2 a^2 + 2 + fa\sqrt{f^2 a^2 + 4}}{2}.$$ By Lemma 1.3 (iii) for $d = f^2 a^2 + 4$, $\varepsilon_0 = \frac{fa + \sqrt{f^2 a^2 + 4}}{2}$ is the fundamental unit by RD type (ii). Thus by $\varepsilon_0{}^2 = \left(\frac{fa + \sqrt{f^2 a^2 + 4}}{2}\right)^2 = \frac{(f^2 a^2 + 2) + fa\sqrt{f^2 a^2 + 4}}{2}$, we have $\varepsilon_0 = \sqrt{\varepsilon_D}$.

Remark 2.5 Let $D = 32 = 2^2 \cdot 8$ be the product of a square of the prime conductor $f = 2$ and the discriminant $d = 8$. The field $Q(\sqrt{D}) = Q(\sqrt{d})$ does not belong to RD type (ii), even though it holds that $D = 6^2 - 4; -6 < -4 < 6$ and $-4 \mid 4 \cdot 6$ and $d = 2^2 + 4, 4 \mid 4 \cdot 2$ because D is not square free and $4 = r > n = 2$. While $D = (6-1)^2 + 7$; but $7 \nmid 4 \cdot 5$ and hence we have $D = (6-2)^2 + r_{-2} = 4^2 + 16, 16 \mid 4 \cdot (6-2)$. Then by Theorem 2.1 and Theorem 2.4 it holds that

$$\varepsilon_D = \frac{2(6-2)^2 + 16 + 2(6-2)\sqrt{32}}{16}$$

$= 3 + 2\sqrt{2} = \varepsilon_0^{\,2} = \varepsilon_+$ with $\varepsilon_0 = \frac{2+\sqrt{8}}{2}$, where ε_+ denotes the totally positive fundamental unit of K and in this case $\varepsilon_+ \in Z\left[1, fw\right]$ with $f = 2, w = \frac{8+\sqrt{8}}{2}$.

Thus we understand that the GRD type involves a much more general assertion than the RD type.

3. LOWER ESTIMATE FOR THE CLASS NUMBERS OF CERTAIN REAL QUADRATIC FIELDS

Yokoi and the second author gave an explicit expression of the fundamental units of real quadratic field district from RD type, respectively (Nakahara, 1970, Yokoi, 1970). In this section we shall evaluate the class numbers of the real quadratic fields containing the fundamental units with norm -1 or +1. For the proof of the density of such fields we apply slight modification of Lemma 8.22 (Narkiewicz, 2004).

Hasse determined a lower bound inequality for the class number of real quadratic field with genus one (Hasse,1965). Theorem 3.3 and Theorem 3.6 involve lower estimates of the class numbers of the fields intersecting trivially to the fields of RD type with genus number not always being equal to one.

Lemma 3.1 (Hasse, 1965).*Let K be a real quadratic field $Q(\sqrt{D})$ of non-square number D and the fundamental unit $\varepsilon_D = \frac{t+u\sqrt{D}}{2} > 1$. If Pell's equation $\frac{x^2 - Dy^2}{4} = \pm p^e$ for a prime number p is solvable, then the following inequality holds:*

$$p^e \geqq s = \begin{cases} \frac{t}{u^2}, & if \quad N(\varepsilon_D) = -1, \\ \frac{t-2}{u^2}, & if \quad N(\varepsilon_D) = 1 \end{cases}$$

with $\left(\frac{D}{p}\right) = 1$ and e is the order of the ideal class $[\underline{p}]$ with $p \cong \underline{p} \cdot \underline{p}^{\sigma}$. Here $\left(\frac{D}{p}\right)$ denote the Kronecker symbol and for a number α and an ideal A in $K, \alpha \cong \mathsf{A}$ denotes that both sides are equal to each other as ideals.

Proposition 3.2 *There exist infinitely many real quadratic fields $Q(\sqrt{d})$ with the field discriminants $d = u^2m^2 \pm 2am + b$ and $a^2 - u^2b = -4$, where $u = \prod_i p_i$ with $p_i = 2$ or each prime factor $p_i \equiv 1$ (mod 4) $i \geq 1$, and with the fundamental unit ε_d such that $\varepsilon_d = \frac{t + u\sqrt{d}}{2}$ satisfying $t = u^2m \pm a$ and $N(\varepsilon d) = -1$. Then the fields $Q(\sqrt{d})$ are not of the RD, ERD nor GRD type if and only if $u \neq 1$ or 2.*

Proof: Let $d = u^2m^2 \pm 2am + b$ be the field discriminant with $u^2b = a^2 + 4$. Then the explicit formula of the fundamental units ε_d with $N(\varepsilon_d) = -1$ is given except for finitely many d by Theorem 1 in (Nakahara,1970). If $d - 1 = s^2$, then $d - 1 - s^2 = u^2m^2 \pm 2am + b - 1 - s^2 = 0$, which is denoted by $f(m)$. Suppose that $f(m)$ is reducible over Q. Then $a^2 - u^2(b - 1 - s^2) = u^2s^2 + u^2 - 4$ is equal to a square number t^2 with $u^2s^2 - t^2 = -u^2 + 4$. Since $-u^2 + 4$ is a constant, then it follows that $\#\left\{(s,t); (us - t)(us + t) | u^2 - 4\right\} < \infty$ for a fixed u. Then the quadratic fields $Q(\sqrt{d})$ do not coincide with RD type (i) except for finitely many numbers of d. The same assertion holds for the case of $d - 4 = s^2$. Then, if d is the field discriminant with $\varepsilon_d = \frac{(u^2m \pm a) + u\sqrt{d}}{2}$ and

$N(\varepsilon_d) = \frac{(u^2m \pm a)^2 - u^2d}{4} = -1$, any $Q(\sqrt{d})$ does not included in F_{RD}, F_{ERD} nor F_{GRD} except for finitely many d, because it holds that for $D = u^2d = (u^2m \pm a)^2 + 4$,

by $u \geq 2, Q(\sqrt{d}) \notin F_{RD} \cup F_{ERD}$

and by $N(\varepsilon_d) < 0, Q(\sqrt{d}) \notin F_{GRD}$. Now consider $u^2d = u^4m^2 \pm 2u^2am + u^2b = (u^2m \pm a)^2 - a^2 + u^2b$, which is denoted by $f_u(m)$. Here $f_u(m) = (u^2m \pm a)^2 + 4 \equiv 1$ (mod 4) holds, if m and a have the district parity modulo 2. Let $A_p(T) = \#\left\{m \leqq T, p^2 \left| \frac{1}{u^2} f_u(m) \right.\right\}$ for a prime p and $A(T)$

$= \#\left\{ m \leqq T; \frac{1}{u^2} f_u(m) \text{ is a square free number} \right\}$.

Then $A(T) \geqq T - \sum_{5 \leqq p < \sqrt{f_u(m)}} A_p(T)$ by $\left(\frac{-4}{3}\right) = -1$.

Since

$A_p(T) \leqq \left(\left[\frac{T}{p^2}+1\right]\right)\sum_{x^2\equiv -4 (\mathrm{mod}\, p^2)} 1 \leqq \frac{2T}{p_2}+2,$

then we have

$A(T) \geqq T - 2T\sum_{5\leqq p}\frac{1}{p^2} - 2\sum_{5\leqq p} 1$

$= \left(1-2\sum_{5\leqq p}\frac{1}{p^2}\right)T + o(T),$ because of the prime number theorem, $\sum_{5\leqq p<T} 1 \sim \frac{T}{\log T}$ holds, where $g(T)=o(h(T))$ and $g(T) \sim h(T)$ means that $\lim_{T\to\infty}\frac{g(T)}{h(T)}=0$ and $\lim_{T\to\infty}\frac{g(T)}{h(T)}=1$, respectively. Here $[\alpha]$ denotes the greatest integer not exceeding α. Since

$\sum_{5\leqq p}\frac{1}{p^2} < \sum_{1\leqq n}\frac{1}{n^2} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$

$< \frac{\pi^2}{6} - \frac{17}{12} \leqq \frac{1}{4}$ holds by virtue of $\zeta(2)=\sum_{1\leqq n}\frac{1}{n^2}=\frac{\pi^2}{6}$, we obtain that $A(T) \geqq \left(1-\frac{2}{4}\right)T + o(T) > \frac{1}{2}T + o(T)$. Next, if m and a have the same parity modulo 2, e.g., for $m=2m_1$ and $a=2a_1$ put

$\tilde{f}_u(m) = (u^2m \pm a)^2 + 4 = 4 f_u(m_1) = 4(u^2 m_1 \pm a_1)^2 + 1).$

Then we have the same conclusion of $f_u(m_1)$ as in the case of $f_u(m)$.

Finally we determine a behavior of the class number of $Q(\sqrt{d})$, where $d = u^2m^2 \pm 2am + b$ with $u^2b = a^2 + 4$ and the density of such fields.

Theorem 3.3 *Let $d = u^2m^2 \pm 2am + b$ be the field discriminant with $u^2b = a^2 + 4$ and F be the family of such real quadratic fields $Q(\sqrt{d})$. Then for any field $Q(\sqrt{d})$ in F, the fundamental unit ε_d is given by*

$$\varepsilon_d = \begin{cases} \frac{(u^2m \pm a) + u\sqrt{d}}{2} & d \equiv 1 (\mathrm{mod}\ 4) \\ \frac{(u^2m \pm a)}{2} + u\sqrt{\frac{d}{4}}, & d = 8d_1 \end{cases}$$

where d_1 is the odd field discriminant and $N\varepsilon_d = -1$. Let h_d be the class number of $Q(\sqrt{d})$. Then it holds that for a prime number p with $\left(\frac{d}{p}\right) = 1$

$$h_d > \frac{\log \frac{u^2m \pm a}{u^2}}{\log p}$$

and there exists $Q(\sqrt{d})$ in F such that $h_d \geqq M$ for any $M > 1$, namely $\limsup_{Q(\sqrt{d})\in F} h_d = \infty$.

Proof: Since p is completely decomposed prime number in $Q(\sqrt{d})$, we obtain by Lemma 3.1 $h_d > \frac{\log \frac{u^2m \pm a}{u^2}}{\log p}$. When we choose m such that for any $M > 1$,

$$\text{namely } \frac{u^2m \pm a - (1 + N(\varepsilon_d))}{u^2} > p^M,$$

$$\log \frac{u^2m \pm a - (1 + N(\varepsilon_d))}{u^2} > M \log p, \text{ that is}$$

$$m > \frac{u^2 p^M \mp a + (1 + N(\varepsilon_d))}{u^2} = p^M + \frac{\mp a + (1 + N(\varepsilon_d))}{u^2},$$

we deduce that $h_d > M$. Therefore we construct a family of real quadratic fields which contains infinitely many quadratic fields with class number greater than an arbitrary positive constant.

Example 3.4 Let $d = 25m^2 \pm 22m + 5$ with an even m and $5^2 \cdot 5 = 11^2 + 4$ in Theorem 3.3. Then the fundamental unit of $Q(\sqrt{d})$ is given by

$$\varepsilon_d = \frac{(25m \pm 11) + 5\sqrt{d}}{2}, d \equiv 1 (\mathrm{mod}\, 4), N(\varepsilon_d) = -1$$

Let $d = 4597$ with $m = 14$, then by the fundamental unit $e_d = \frac{339 + 5\sqrt{4597}}{2}$ and $(\frac{d}{3}) = 1$, we have $h_d > \frac{\log \frac{25 \cdot 14 - 11}{25}}{\log 3} > 2.37$. In fact, we have $h_d = 3$.

Example 3.5 Let $d = 25m^2 \pm 28m + 8$ with an even m and $5^2 \cdot 8 = 14^2 + 4$ in Theorem 3.3. Then the fundamental unit of $Q(\sqrt{d})$ is given by

$$\varepsilon_d = \frac{(25m \pm 14)}{2} + 5\sqrt{\frac{d}{4}}, 8 \mid d, N(\varepsilon_d) = -1$$

Let $d = 33416 = 2^3 \cdot 4177$ with $m = 36$, then by the fundamental unit $457 + 5\sqrt{8354}$ and $(\frac{d}{5}) = 1$, we have $h_d = \frac{\log \frac{25 \cdot 36 + 14}{25}}{\log 5} > 2.22$. In fact, we have $h_d = 8$.

From Theorem 2.1 and Lemma 3.1, a lower bound of the class number of a GRD type can be deduced.

Example 3.6 Let $D = 4053 = (-3)\cdot(-7)\cdot 193 = 64^2 - 43,43 \mid 4\cdot 64$. By $D = (64-1)^2 + 84,\ 84 \mid 4\cdot(64-1)$, then $Q(\sqrt{4053})$ is of a GRD and an ERD type. The fundamental unit ε_D of $Q(\sqrt{4053})$ is $\frac{191+3\sqrt{4053}}{2}$ with $N(\varepsilon_D) = +1$ and $\left(\frac{4053}{11}\right) = 1$. Then $h_{ij} > \frac{\log 84 - 2\log 2}{\log 11} > 1.26$. By GauB genus theory $2^{3-1} \mid h_+(4053)$ holds, which implies that $h(D) = h_+(D)/2$ is at least 2. In fact, the class number $h(D)$ is equal to four, because we have four classes of reduced numbers.

$$\mathfrak{R}_1 = \left\{\frac{63+\sqrt{4053}}{2}, \frac{63+\sqrt{4053}}{42}\right\},$$

$$\mathfrak{R}_2 = \left\{\frac{63+\sqrt{4053}}{6}, \frac{63+\sqrt{4053}}{14}\right\},$$

$$\mathfrak{R}_3 = \left\{\frac{15+\sqrt{4053}}{58}, \frac{43+\sqrt{4053}}{38}, \frac{33+\sqrt{4053}}{78}, \frac{45+\sqrt{4053}}{26}, \frac{59+\sqrt{4053}}{22}, \frac{51+\sqrt{4053}}{66}\right\},$$

$$\mathfrak{R}_4 = \left\{\frac{15+\sqrt{4053}}{66}, \frac{51+\sqrt{4053}}{22}, \frac{59+\sqrt{4053}}{26}, \frac{45+\sqrt{4053}}{78}, \frac{33+\sqrt{4053}}{38}, \frac{43+\sqrt{4053}}{58}\right\}$$

We confirm that the class group A_K / P_K is cyclic, where A_K and P_K denote the group of fractional ideals of a real quadratic field $K = Q(\sqrt{d})$ and the subgroup of principal fractional ideals of K, respectively. For an ideal $\mathfrak{I} = Z\left[j, \frac{r+\sqrt{D}}{2}\right]$ of A_K with $j, r \in Z$, let $\left\{j, \frac{r+\sqrt{D}}{2}\right\}$ denote a canonical basis i.e., $j \mid N\left\{j, \frac{r+\sqrt{D}}{2}\right\}$. Since the principal ideal class $[A]$ with $A = Z\left[1, \frac{63+\sqrt{4053}}{2}\right]$ corresponds to the first class $\mathfrak{R}_1$ and the order of the second class (B_3) corresponds to $\mathfrak{R}_2$ is equal to two by the composition $B_3 B_7 = B_{21} = Z\left[21, \frac{63+\sqrt{4053}}{2}\right]$ of $B_3 = Z\left[3, \frac{63+\sqrt{4053}}{2}\right]$ and $B_7 = Z\left[7, \frac{63+\sqrt{4053}}{2}\right]$ with $B_{21} \in [A]$. Then it is deduced that the square of the third class $[C_{13}] = Z\left[13, \frac{45+\sqrt{4053}}{2}\right]$ corresponds to R_3 isequal to $[B_3]$. In fact, we choose two ideals C_{13} and $C_{11} = Z\left[11, \frac{59+\sqrt{4053}}{2}\right]$. Thus the square class $[C_{13}]^2$ contains the product ideal $C_{13}C_{11} = Z\left[143, \frac{u+\sqrt{4053}}{2}\right]$, where u satisfies $u \equiv 45 \pmod{13}$ and $u \equiv 59 \pmod{11}$, and hence $u \equiv -7 \pmod{2\cdot 13\cdot 11}$. Put $\eta_0 = \frac{-7+\sqrt{4053}}{2\cdot 13\cdot 11} = 0 + \frac{1}{\eta_1}$ with $\eta_1 = \frac{7+56+\sqrt{4053}}{2\cdot 7}$. By $Z\left[7, \frac{7+\sqrt{4053}}{2}\right] = Z\left[7, \frac{7+56+\sqrt{4053}}{2}\right]$, the irrational number $\frac{63+\sqrt{4053}}{14}$ belongs to $\mathfrak{R}_2$. Therefore A_K / P_K is cyclic group of the order 4.

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