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Invariant Results of Viscoelastic Fluid Flow without Porous space in a Pipe using Oldroyd–B Constitutive Model: Analysis of Velocity

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Abstract In the field of fluid dynamics, investigation of fluid flows through porous media and mathematical modeling is very difficult work for solving the various challenging problems in applied sciences. The Lie group theory is important approach for solving the differential equations. In this paper, the system of flow equations containing of the conservation of mass and momentum transport equation joined with constitutive model of the Oldroyd–B which is solved using the Lei- group method by finding the symmetries, so that accurate invariant solution in the analysis of velocity is acquired.

Keywords: Viscoelastic Flow in a pipe without porous space, Darcy-Brinkman model, Oldroyd–B constitutive model, Symmetry method, Results with Velocity

1 INTRODUCTION

Fluid mechanics is one of the most interesting and beautiful subject of science and daily interaction with fluids such as water and air in the atmosphere, construct this as one of the most stirring topic for researchers. It is always been exciting to carry out the study fluids flow through porous media. Major advances have been made in developing the momentum equation that governs the fluid flow in porous media, starting from the Darcy law to the generalized Darcy model. A generalized Darcy model for the fluid flow through porous medium was developed to explanation for inertial effects, boundary effects, and variable porosity medium. These effects are included by using the general fluid flow model known as Darcy-Brinkman-Forchheimer model. The solution of differential equations governing the compressible or incompressible fluids, Newtonian fluids or non-Newtonian fluids is sensitively involved significant to interest in the literature. The modeling of flow through dense molecular structure such as polymer solutions, slurries, blood, pastes and paints related with the research of non-Newtonian fluids. Both elastic properties like solids and viscous properties like liquids have been displayed in these materials and understanding of their difficult activities is crucial in many industrial applications. Flows of Newtonian and non-Newtonian fluids connected with some important exploration are prepared by way of Ariel *et al.* (2006), Abel-Malek *et al.* (2002), Chen *et al.* (2006), Bird *et al.* (1981, 1983, 1987), Fetecau and Fetecau (2005. 2006), Rajagopal and Gupta (1984), Rajagopal and Na (1985), and Wafo- (2005), As nature of modeling for flow of viscoelastic fluids there is not only one constitutive equation available in the literature, which describes the properties of flow associated with all non-Newtonian fluids. For this reason, a range of models have been suggested and among those models, power-law and fluids of differential type have obtained a great arrangement of concentration. There is simple proportionality involving the viscosities within unlike kinds of deformation, and zero normal stress differences within simple shear flow in Newtonian fluids, which are generally characteristics of these fluids by containing shear- and time-independent viscosity, investigation are extended by Owens and Phillips (2002), Sochi (2009, 2010),. van Os and Phillips.(2004), and others. There are several constitutive equations which predict qualitatively the behaviour of some of the material functions, requiring only a small number of free constants. Some of these constitutive equations, however, are complicated. Here viscoelastic behaviour will be modelled by the Oldroyd-(Oldroyd (1950) and Phan-thien/Tanner (1977) differential constitutive models.

The basic purpose is to present the Lei- group analysis for viscoelastic fluid flows to obtained the solution to the system of PDE's comprising the continuity, momentum and constitutive equations, under appropriate initial and boundary conditions.This paper presents the transient hydrodynamics behaviour of the Viscoelastic flow in pipes without porous media related with Darcy-Brinkman model, separated and the system of equations of flow comprises of the conservation of

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mass and conservation of momentum transport coupled with the Oldroyd–B constitutive model for analysis of velocity. The research is discussed by means of analytical and numerical solutions of the problems of governing equations associated with initial and boundary conditions, occurring in the knowledge of viscoelastic flow in pipes without porous media related with velocity.

The solutions of problem are obtained adopting the transformations group theoretic approach and the analytical solution of the equations is found under the symmetries for the PDE's through Lie group method. The one-parameter group transformation reduces one independent variable from the number of independent variables and the governing PDE's system are reduced in to ODEs system with suitable boundary conditions corresponding to the symmetry group, so that correct invariant solutions can be obtained. Lie-Group theory of ODE's and PDE's as a scientific branch produced from exertions of the excellent mathematician Lie of the nineteenth century (1842–1899) and developed by Olver, (1986), Ibragimov, (1999), Bluman and Kumei (1989), and others researchers. After long time, Ovsiannikov (See Ibragimov, 1999) awakened interest to group analysis and show that the properties explanation of differential equations in terms of acceptable groups is uniformly well-organized for constructions of exact solutions of PDEs and also for research of differential equations of mathematical physics and mechanics. The problem is to find and make use of accepted Lie point symmetries algebra. Numerical predictions of a system are resolute adopting Mathematica solver ND-Solve.

Formulations of problem are related in section 2. Section 3 associated with viscoelastic flow in pipes without porous space and its solutions; section 3.1 connected with study state solution for non homogeneous equation; section 3.2 linked with Lie-point symmetries of the system of PDE's (13-i & iii) of viscoelastic flow without porous space in pipes. Analytical solutions corresponding to the generator $X_1-\beta X_2$ presented in section 3.3, section 3.4 connected with result of PDEs (13-ii). Section 4 associated with analysis of Velocity. As section 4.1 concerned with Exact Solution of Velocity and also graphed discussed. In section 4.2 steady state solution is planted and discussed. As section 4.3 associated with numerical results of PDE's (7, 8 & 9) of viscoelastic flow without porous media in pipes and in section 5, conclusions are presented.

2 PROBLEM FORMULATIONS

Consider the incompressible laminar flow of viscoelastic fluid in a pipe filled with porous medium. A polar coordinate system is applied with radius-axis vertically upward. The system of governing equations of flow comprises of the conservation of mass and conservation of momentum transport coupled with the Oldroyd–B constitutive model. The viscoelastic fluid flow in the course of porous medium is supposed to possess homogeneous and isotropic. For unidirectional flow velocity field is known as $\bar{u}=\left(u(r,t),0,0\right)$; wherever the above sense of velocity mechanically satisfies the incompressibility state. The generalized Darcy–Brinkman model has been employed for the momentum equation and in the absence of body force,; continuity equation, generalised equation of momentum through porous media and the Oldroyd–B equation defines the stresses of viscoelastic in the fluid flow in vectorial form may be written in the following form:

$$\nabla.\bar{u}=0 \quad (1)$$

$$\frac{\rho}{\varepsilon}\frac{\partial\bar{u}}{\partial t}=\frac{1}{r}\nabla([\frac{\mu_2}{\varepsilon}r\,\underline{\underline{d}}]+\tau)-\nabla p-\rho\bar{u}.\nabla\bar{u}-\frac{\mu}{K}\bar{u}\ ,(2)$$

where $\frac{\partial}{\partial t}$ is a temporal derivative with respect to time t, $.\bar{u}$ is defined for the field of velocity vector. ∇ is the spatial differential operator, p is the isotropic fluid pressure (per unit density) and t is the time, ρ and μ is the fluid density and total viscosity of viscoelastic fluid respectively, τ is the extra stress tensor, $\underline{\underline{d}}$ is the rate–of–strain tensor, μ_2 is indicated for the Newtonian solvent viscosity, p is the isotropic fluid pressure, the acceleration co–efficient tensor is assumed to be $1/\varepsilon$ and ε is porosity of porous media. and the intrinsic permeability within porous media is identified with K.

The constitutive equation of Oldroyd–B model describes the stresses of viscoelastic in the fluid flow can be given as under as: $\lambda\frac{\partial\tau}{\partial t}=[2\mu_1\underline{\underline{d}}]-\tau-\lambda\{\bar{u}.\nabla\tau-\nabla\bar{u}.\tau-(\nabla\bar{u})^T.\tau\}$ (3)

Where the rest time for the fluid of viscoelastic is defined by λ and μ_1 is indicated for viscoelastic solute viscosity. As total viscosity is $\mu=\mu_1+\mu_2=1$ and is taken constant.

The equations are obtained which govern the unsteady unidirectional flow of viscoelastic fluid through porous media adopting Oldroyd–B constitutive model. For unidirectional flow the velocity field is $\bar{u}=\left(u(r,t),0,0\right)$; here the description of velocity automatically presents pleasure to the incompressibility state. The derivation of such equations by employing the momentum transport equation of viscoelastic fluid and Oldroyd–B constitutive equations assuming constant pressure gradient and may be expressed in the absence of body force as follows:

$$\mathrm{Re}\,\frac{\partial u}{\partial t}=1+\mu_2\frac{\partial^2 u}{\partial r^2}+\frac{\mu_2}{r}\frac{\partial u}{\partial r}+\frac{\partial \tau_{12}}{\partial r}+\frac{\tau_{12}}{r}-\frac{1}{Da}u \quad \text{(4-i)} \qquad We\,\frac{\partial \tau_{11}}{\partial t}=2\,We\,\tau_{12}\frac{\partial u}{\partial r}-\tau_{11} \quad \text{(4-ii)}$$

$$We\,\frac{\partial \tau_{12}}{\partial t}=\mu_1\frac{\partial u}{\partial r}-\tau_{12} \quad \text{(4-iii)}$$

To complete the well posed problem requirement, it is necessary to set initial and boundary conditions. So initial conditions and boundary conditions are taken as:

$$u(t,1)=0,\ \text{and}\ \frac{\partial u}{\partial t}(t,0)=0 \ \text{When } t>0 \ (5) \quad u(0,r)=\tau_{11}(0,r)=\tau_{12}(0,r)=0 \quad \text{When } 0<r<1 \ (6)$$

Where u and τ are dimensionless velocity and dimensionless stress tensor, *r* is radial coordinates, *t* is the time using for non-dimensional and the dimensionless Reynolds number (*Re*), Weissenberg number (*We*) and Darcy's number (*Da*) are defined as

$$\mathrm{Re}=\frac{R\,\rho\,Vc}{\mu},\quad We=\frac{\lambda\,Vc}{R},\quad Da=\frac{K}{\varepsilon\,R^2}.$$

Hence K is the adapted permeability concern with the porous medium using for non-dimensional. As *R* is a radius of the pipe and Vc is used for the feature velocity supposed since reference redial velocity $Vc=\dfrac{\varepsilon R^2\left(-\frac{\partial p}{\partial z}\right)}{\mu}$

3 Viscoelastic Flow in Pipes without Porous space and its Solutions

As Darcy's number Da approaches to infinity, or the last Darcy's term *Da* vanishes (i.e. $Da\to\infty$), then the system (4) is called viscoelastic flow in pipes without porous space is taken s under.

$$\mathrm{Re}\,\frac{\partial u}{\partial t}=1+\mu_2\frac{\partial^2 u}{\partial r^2}+\frac{\mu_2}{r}\frac{\partial u}{\partial r}+\frac{\partial \tau_{12}}{\partial r}+\frac{\tau_{12}}{r}-\frac{1}{Da}u \quad \text{(7-i)} \qquad We\,\frac{\partial \tau_{11}}{\partial t}=2\,We\,\tau_{12}\frac{\partial u}{\partial r}-\tau_{11} \quad \text{(7-ii)}$$

$$We\,\frac{\partial \tau_{12}}{\partial t}=\mu_1\frac{\partial u}{\partial r}-\tau_{12} \quad \text{(7-iii)}$$

Subject to same boundary and initial conditions are,

$$u(t,1)=0,\ \text{and}\ \frac{\partial u}{\partial t}(t,0)=0 \ \text{When } t>0 \ (8) \quad u(0,r)=\tau_{11}(0,r)=\tau_{12}(0,r)=0 \quad \text{When } 0<r<1 \ (9)$$

3.1 Steady State Solution for non Homogeneous Equation

Some problems involving non–homogeneous equations can be solved by means of a change of dependent variable and to find the steady state solution, so for this consider

$$u(t,r)=v_1(t,r)+\varphi_1(r),\ \tau_{11}(t,r)=v_2(t,r)+\varphi_2(r)\ \text{and}\ \tau_{12}(t,r)=v_3(t,r)+\varphi_3(r) \quad (10)$$

putting these values in Equation (7-i, ii, iii) and separating the like terms of independent variables, gives the two systems of equations. The first system is

$$\mu_2\varphi_1''(r)+\frac{\mu_2}{r}\varphi_1'(r)+\varphi_3'(r)+\frac{\varphi_3(r)}{r}=0,\quad \varphi_2(r)=2\,We\,\varphi_3(r)\varphi_1'(r),\quad \varphi_3(r)=\mu_1\,\varphi_1'(r) \quad (11)$$

Subject to boundary condition: $\varphi_1(1)=0$ and $\varphi_1'(0)=0$ and after solving and integrate and applying the boundary conditions, result of above system admit the steady–state solutions as below:

$$\varphi_1(r)=\frac{1}{4}\left(1-r^2\right) \quad \text{(12-i)} \qquad \varphi_2(r)=\frac{We\,\mu_1}{2}r^2 \quad \text{(12-ii) and} \qquad \varphi_3(r)=\frac{-\mu_1}{2}r \quad \text{(12-iii)}$$

and second system which is PDE's to determine $v_1(t,r), v_2(t,r)$ and $v_3(t,r)$, the new boundary value problem is obtained as

$$\mathrm{Re}\,\frac{\partial v_1}{\partial t}=\mu_2\frac{\partial^2 v_1}{\partial r^2}+\frac{\mu_2}{r}\frac{\partial v_1}{\partial r}+\frac{\partial v_3}{\partial r}+\frac{v_3}{r} \quad \text{(13-i)} \qquad We\,\frac{\partial v_2}{\partial t}=2We\,(v_3-\frac{\mu_1}{2}r)\frac{\partial v_1}{\partial r}-r\,We\,v_3-v_2 \quad \text{(13-ii)}$$

$$We\,\frac{\partial v_3}{\partial t}=\mu_1\frac{\partial v_1}{\partial r}-v_3 \quad \text{(13-iii)}$$

Subject to initial and boundary conditions are taken as,

$$v_1(t,1)=0 \quad \text{(14-i)}, \qquad \frac{\partial v_1(t,0)}{\partial r}=0 \quad \text{(14-ii)} \qquad t>0 \quad \text{and} \quad v_1(0,r)=\frac{-1}{4}(1-r^2) \qquad \text{(15-i)}$$

$$v_2(0,r)=\frac{-We\,\mu_1}{2}r^2 \quad \text{(15-ii)} \qquad v_3(0,r)=\frac{\mu_1}{2}r \qquad \text{(15-iii)}$$

3.2 Symmetry Analysis for the PDE's System (13-i &-iii) of Viscoelastic Flow in a Pipe without Porous space

Lie group method is powerful method in obtaining analytical solutions of differential equations. Once symmetry Lie algebra of the differential equation is known, it can be used in the investigation of transformations that will reduce the equation to simpler form. Hence derivatives of equations (13-*i*) and (13-*iii*) are linked each other. So in this section, symmetry conditions and method for finding the Lie point symmetries of the equations (13-*i*) and (13-*iii*) are introduced. The generator

$$X=\phi(t,r,v_1,v_3)\frac{\partial}{\partial t}+\xi(t,r,v_1,v_3)\frac{\partial}{\partial r}+\eta^1(t,r,v_1,v_3)\frac{\partial}{\partial v_1}+\eta^2(t,r,v_1,v_3)\frac{\partial}{\partial v_3} \qquad (16)$$

is the Lie point symmetry generator for governed PDE's system (13-*i*) and (13-*iii*) if,

$$X^{[2]}\left(\mu_2 v_{1rr}+\frac{\mu_2}{r}v_{1r}+v_{3r}+\frac{v_3}{r}-\operatorname{Re} v_{1t}\right)\Bigg|_{(13-i\,\&\,iii)}=0, \qquad X^{[1]}(We\ v_{3t}-\mu_1 v_{1r}+v_3)\Big|_{(13-i\,\&\,iii)}=0$$

where first and second extended infinitesimal generator of X are

$$X^{[1]}=X+\eta_t^{1[1]}\frac{\partial}{\partial u_t}+\eta_r^{1[1]}\frac{\partial}{\partial v_{1r}}++\eta_t^{2[1]}\frac{\partial}{\partial v_{3t}}+\eta_r^{2[1]}\frac{\partial}{\partial v_{3r}} \qquad X^{[2]}=X^{[1]}+\eta_{rr}^{1[2]}\frac{\partial}{\partial v_{1rr}}+\ldots. \qquad (17)$$

In which $\eta_t^{1[1]}$, $\eta_r^{1[1]}$, $\eta_t^{2[1]}$, $\eta_r^{2[1]}$, $\eta_{rr}^{1[2]}$ are written as

$$\eta_r^{1[1]}=D_r\eta^1-v_{1t}\,D_r\phi-v_{1r}\,D_r\xi; \qquad \eta_t^{2[1]}=D_t\eta^2-v_{3t}\,D_t\phi-v_{3r}\,D_t\xi;$$

$$\eta_r^{2[1]}=D_r\eta^2-v_{3t}\,D_r\phi-v_{3r}\,D_r\xi; \qquad \eta_{rr}^{1[2]}=D_r\eta_r^{1[1]}-v_{1tr}\,D_r\phi-v_{1rr}\,D_r\xi. \qquad (18)$$

Where D_t and D_r are the total derivative operators given as

$$D_t=\frac{\partial}{\partial t}+v_{1t}\frac{\partial}{\partial v_1}+v_{1tr}\frac{\partial}{\partial v_{1t}}+v_{1tt}\frac{\partial}{\partial v_{1t}}+v_{3t}\frac{\partial}{\partial v_3}+v_{3r}\frac{\partial}{\partial v_{3tr}}+v_{3tt}\frac{\partial}{\partial v_{3t}}+v_{3tr}\frac{\partial}{\partial v_{3r}}+\ldots\ldots\ldots\ldots,$$

$$D_r=\frac{\partial}{\partial r}+v_{1r}\frac{\partial}{\partial v_1}+v_{1tr}\frac{\partial}{\partial v_{1t}}+v_{1rr}\frac{\partial}{\partial v_{1r}}+v_{3r}\frac{\partial}{\partial v_3}+v_{3t}\frac{\partial}{\partial v_{3tr}}+v_{3rr}\frac{\partial}{\partial v_{3r}}+v_{3tr}\frac{\partial}{\partial v_{3t}}+\ldots\ldots\ldots\ldots, \qquad (19)$$

In the operator X, according to Lie's theory, the unknown functions ϕ, ξ and η are taken independent of the derivatives of the primitive variables v_1 and v_3. As

$$X^{[2]}\left(\mu_2 v_{1rr}+\frac{\mu_2}{r}v_{1r}+v_{3r}+\frac{v_3}{r}-\operatorname{Re} v_{1t}\right)\Bigg|_{(13-i\,\&\,iii)}=0 \Rightarrow -\frac{1}{r^2}(\mu_2 v_{1r}+v_3)\xi+\frac{1}{r}\eta^2-\operatorname{Re}\eta_t^{1[1]}+\frac{\mu_2}{r}\eta_r^{1[1]}+\eta_r^{2[1]}+\mu_2\eta_{rr}^{1[2]}\Bigg|_{(13-i\,\&\,iii)}=0 \quad \text{(I)}$$

$$X^{[1]}(We\ v_{3t}-\mu_1 v_{1r}+v_3)\Big|_{(13-i\,\&\,iii)}=0 \Rightarrow \eta^2-\mu_1\eta_r^{1[1]}+We\,\eta_t^{2[1]}\Big|_{(13-i\,\&\,iii)}=0 \qquad \text{(II)} \qquad (20)$$

Where $v_{1t}, v_{1r}, v_{1rr}, v_{3t}, v_{3r}$, etc, .are partial derivatives.

Where $X^{[1]}$, $X^{[2]}$ and ($\eta_t^{1[1]}$, $\eta_r^{1[1]}$, $\eta_{rr}^{1[2]}$, $\eta_t^{2[1]}$, $\eta_r^{2[1]}$) are described in the relations (18 & 19). In the above equations, the unidentified functions ϕ, ξ, η^1 and η^2 are independent for the differentials of v_1 and v_3. Thus separating w. r. to the differentials of v_1 and v_3 and powers of the differentials of v_1 and v_3 leads to the two simplified over determined systems of PDE's and after solving these two over determined systems of linear PDE's, solution of the two over resolved systems gives rise to the values of the functions ϕ, ξ, η^1 and η^2 are given as:

$$\phi=c_1, \quad \xi=0, \quad \eta^1=c_2 v_1+g_1(t,r), \quad \eta^2=c_2 v_3+g_2(t,r) \qquad (21)$$

Hence c_i are arbitrary constants of integration and $g_1(t,r)$, $g_2(t,r)$ are an arbitrary functions of the PDEs of the form

$$\mathrm{Re}\frac{\partial g_1}{\partial t}=\mu_2\frac{\partial^2 g_1}{\partial r^2}+\frac{\mu_2}{r}\frac{\partial g_1}{\partial r}+\frac{\partial g_2}{\partial r}+\frac{g_2}{r} \quad \text{and} \quad We\frac{\partial g_2}{\partial t}=\mu_1\frac{\partial g_1}{\partial r}-g_2 \tag{22}$$

Thus the symmetry Lie algebra of the system of equations (13-i & iii) is two-dimensional and defined by the following generators

$$X_1=\frac{\partial}{\partial t}, \qquad X_2=v_1\frac{\partial}{\partial v_1}+v_3\frac{\partial}{\partial v_3} \quad \text{and} \quad X_m=g_2(t,r)\frac{\partial}{\partial v_1}+g_3(t,r)\frac{\partial}{\partial v_3} \tag{23}$$

where 'm' are non negative integer.

3.3 Analytical Solutions corresponding to the generator $X_1 - \beta X_2$

$$X=X_1-\beta X_2=\frac{\partial}{\partial t}-\beta v_1\frac{\partial}{\partial v_1}-\beta\, v_3\frac{\partial}{\partial v_3}$$

From given generator, the invariant solutions corresponding to *X,* are obtained by solving the characteristic system

$$\frac{dt}{1}=-\frac{dv_1}{\beta\, v_1}=-\frac{dv_3}{\beta\, v_3}$$

The invariant solutions admitted by the operator *X* are given by

$$v_1(t,r)=e^{-\beta t}\psi_1(r), \qquad v_3(t,r)=e^{-\beta t}\psi_3(r)\Big\} \tag{24}$$

Put the above relations (25) into governing equations (13-i) and (13-iii).shows ODE's of the functions $\psi_1(r)$ and $\psi_3(r)$.

$$\mu_2\psi_1''(r)+\frac{\mu_2}{r}\psi_1'(r)+\psi_3'(r)+\frac{1}{r}\psi_3(r)+\beta\mathrm{Re}\psi_1(r)=0 \quad (i) \quad \psi_3(r)=\frac{\mu_1}{(1-\beta We)}\psi_1'(r) \quad (ii) \tag{25}$$

Now, putting $\psi_3(r)$ from (25-ii) into (25-i), the Bessel's differential equation is obtained whose order is zero i.e.

$$\psi_1''(r)+\frac{1}{r}\psi_1'(r)+\alpha^2\psi_1(r)=0 \tag{26}$$

Where $\alpha^2=\dfrac{\beta\ \mathrm{Re}\ (1-\beta\ We)}{(1-\mu_2\ \beta\ We)}$

General solution of the Bessel's differential equation (26) is agreed as

$$\psi_1(r)=A\,J_0(\alpha\, r)+B\,Y_0(\alpha\, r) \tag{27}$$

Where $J_0(\alpha\, r)$ and $Y_0(\alpha\, r)$ are Bessel function of order zero of first and second kind respectively. Of course, equation (26) is singular or $Y_0(\alpha\, r)\to-\infty$ when $r=0$. Physically meaningful solution must be twice continuously differentiable in $0\le r\le 1$. We must obtain $\beta=0$ and equation (27) has only one bounded solution, i.e.

$$\psi_1(r)=A\ J_0(\alpha\ r)$$

Substitute this value of $\psi_1(r)$ in equation (25-ii), then $\psi_3(r)$ is obtained which is given as

$$\psi_3(r)=\frac{-\mu_1\alpha}{(1-\beta We)}AJ_1(\alpha r).$$

After setting the values of functions $\psi_1(r)$ and $\psi_3(r)$, general solutions of the governed PDE's (13-i & iii) are given as under

$$v_1(t,r)=Ae^{-\beta t}J_0(\alpha\, r) \text{ and } v_3(t,r)=\frac{-A\mu_1\alpha e^{-\beta t}}{(1-\beta We)}J_1(\alpha\, r) \tag{28}$$

Applying the boundary condition of equation (14-i), we find that

$$A\,J_0(\alpha)=0. \text{ Hence } A\neq 0, \text{ therefore } J_0(\alpha)=0 \tag{29}$$

Which has infinite number of roots $\alpha=\lambda_n\ (n=1,2,3,.......,\infty)$, so for the graph, we must select the any root of the equation (29). As the Bessel's differential equation (27) is the arrangement of two equations (25-i & ii) which have same boundary conditions, so for functions of time as

$$\alpha^2 = \frac{\beta \operatorname{Re} (1-\beta We)}{(1-\mu_2 \beta We)} \Rightarrow \beta = \frac{1}{2}\left(\frac{1}{We}+\frac{\mu_2\alpha^2}{\operatorname{Re}}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{We}+\frac{\mu_2\alpha^2}{\operatorname{Re}}\right)^2 - \frac{4\alpha^2}{\operatorname{Re} We}}$$

Therefore

$$\beta_1 = \frac{1}{2}\left(\frac{1}{We}+\frac{\mu_2\alpha^2}{\operatorname{Re}}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{We}+\frac{\mu_2\alpha^2}{\operatorname{Re}}\right)^2 - \frac{4\alpha^2}{\operatorname{Re} We}} \quad \& \quad \beta_2 = \frac{1}{2}\left(\frac{1}{We}+\frac{\mu_2\alpha^2}{\operatorname{Re}}\right) - \frac{1}{2}\sqrt{\left(\frac{1}{We}+\frac{\mu_2\alpha^2}{\operatorname{Re}}\right)^2 - \frac{4\alpha^2}{\operatorname{Re} We}}$$

Therefore applying the superposition principle,, equation (24) develops into as under:

$$v_1(t,r) = \sum_{n=1}^{\infty}\left(A_{n1} e^{-\beta_1 t} + A_{n2} e^{-\beta_2 t}\right) J_0(\lambda_n r) \quad \text{(30-i)} \quad \& \quad v_3(t,r) = \sum_{n=1}^{\infty} -\mu_1 \lambda_n \left(\frac{A_{n1} e^{-\beta_1 t}}{(1-\beta_1 We)} + \frac{A_{n2} e^{-\beta_2 t}}{(1-\beta_2 We)}\right) J_1(\lambda_n r) \quad \text{(30-ii)}$$

For the constants, applying the initial conditions (15-i) and (15-iii), when $J_0(\lambda_n) = 0$

$$v_1(0,r) = \sum_{n=1}^{\infty}(A_{n1}+A_{n2}) J_0(\lambda_n r) = \frac{-1}{4}(1-r^2)$$

$$\Rightarrow A_{n1} + A_{n2} = \frac{-\int_0^1 r\,(1-r^2)\, J_0(\lambda_n r)\, dr}{4\int_0^1 J_0^{\,2}(\lambda_n r)\, dr} = \frac{-2}{\lambda_n^3 J_1(\lambda_n)} \quad \text{(31-i)}$$

$$v_3(0,r) = \sum_{n=1}^{\infty} -\mu_1 \lambda_n \left(\frac{A_{n1}}{(1-\beta_1 We)} + \frac{A_{n2}}{(1-\beta_2 We)}\right) J_1(\lambda_n r) = \frac{\mu_1}{2} r \Rightarrow \frac{A_{n1}}{(1-\beta_1 We)} + \frac{A_{n2}}{(1-\beta_2 We)} = \frac{-\int_0^1 r^2\, J_1(\lambda_n r)\, dr}{2\lambda_n \int_0^1 r J_1^2(\lambda_n r)\, dr} = \frac{-2}{\lambda_n^3 J_1(\lambda_n)} \quad \text{(31-ii)}$$

After solving the equations (31-i & ii), we obtain

$$A_{n1} = \frac{2\beta_2 (1-\beta_1 We)}{(\beta_1-\beta_2)\lambda_n^3 J_1(\lambda_n)} \quad \& \quad A_{n2} = \frac{-2\beta_1 (1-\beta_2 We)}{(\beta_1-\beta_2)\lambda_n^3 J_1(\lambda_n)} \quad \text{(32)}$$

Therefore, solution of equations (13-i) and (13-iii) are given as under;

$$v_1(t,r) = \sum_{n=1}^{\infty} \frac{2}{\lambda_n^3 J_1(\lambda_n)} \left(\frac{\beta_2(1-\beta_1 We) e^{-\beta_1 t}}{(\beta_1-\beta_2)} - \frac{\beta_1(1-\beta_2 We) e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right) J_0(\lambda_n r) \quad \text{(33-i)}$$

$$v_3(t,r) = \sum_{n=1}^{\infty} \frac{-2\mu_1}{\lambda_n^2 J_1(\lambda_n)} \left(\frac{\beta_2 e^{-\beta_1 t}}{(\beta_1-\beta_2)} - \frac{\beta_1 e^{-\beta_2 t}}{(\beta_1-\beta_2)}\right) J_1(\lambda_n r) \quad \text{(33-ii)}$$

3.4 Results of Partial differential equation (13-ii)

According to the equation (33-i & ii), Solution of equation(13-ii), is agreed as

$$v_2(t,r) = 2We\,\mu_1\alpha^2 \left(\frac{A_1^2 e^{-2\beta_1 t}}{(1-\beta_1 We)(1-2\beta_1 We)} + \frac{A_2^2 e^{-2\beta_2 t}}{(1-\beta_2 We)(1-2\beta_2 We)} + \frac{A_1 A_2 (2-\beta_1 We-\beta_2 We) e^{-(\beta_1+\beta_2)t}}{(1-\beta_1 We)(1-\beta_2 We)(1-\beta_1 We-\beta_2 We)}\right) J_1^2(\alpha r)$$
$$+ We\,\mu_1 r\alpha \left(\frac{A_1 (2-\beta_1 We) e^{-\beta_1 t}}{(1-\beta_1 We)^2} + \frac{A_2 (2-\beta_2 We) e^{-\beta_2 t}}{(1-\beta_2 We)^2}\right) J_1(\alpha r) + e^{-\frac{1}{We}t}\psi_2(r) \quad \text{(34)}$$

As $J_0(\lambda_n)=0$, so we get $r=\sum_{n=1}^{\infty}\frac{4}{\lambda_n^2} J_1(\lambda_n r)$ and applying the superposition principle and he initial condition, this gives us

$$\psi_2(r)=-\left(\sum_{n=1}^{\infty}(2We\mu_1)^{\frac{1}{2}}\left(\begin{array}{l}\frac{\lambda_n^2A_{n1}^2}{(1-\beta_1We)(1-2\beta_1We)}+\frac{\lambda_n^2A_{n1}A_{n2}(2-\beta_1We-\beta_2We)}{(1-\beta_1We)(1-\beta_2We)(1-\beta_1We-\beta_2We)}\\+\frac{\lambda_n^2A_{n2}^2}{(1-\beta_2We)(1-2\beta_2We)}+\frac{2A_{n1}(2-\beta_1We)}{\lambda_n(1-\beta_1We)^2}+\frac{2A_{n2}(2-\beta_2We)}{\lambda_n(1-\beta_2We)^2}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2-\frac{We\mu_1}{2}r^2$$

After putting the values of A_{n1}, A_{n2} and $\psi_2(r)$, the solution of equation (13-ii) is given as

$$v_2(t,r)=\left(\sum_{n=1}^{\infty}\frac{(8We\mu_1)^{\frac{1}{2}}}{\lambda_n^2J_1(\lambda_n)}\left(\begin{array}{l}\frac{\beta_2^2(1-\beta_1We)(e^{-2\beta_1t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2(1-2\beta_1We)}+\frac{\beta_1^2(1-\beta_2We)(e^{-2\beta_2t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2(1-2\beta_2We)}\\-\frac{\beta_1\beta_2(2-\beta_1We-\beta_2We)(e^{-(\beta_1+\beta_2)t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2(1-\beta_1We-\beta_2We)}\\+\frac{\beta_2(2-\beta_1We)(e^{-\beta_1t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)(1-\beta_1We)}-\frac{\beta_1(2-\beta_2We)(e^{-\beta_2t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)(1-\beta_2We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2-\frac{We\mu_1}{2}r^2e^{-\frac{1}{We}t} \tag{35}$$

Therefore final result of the system (7) show the following solutions

$$u(t,r)=\sum_{n=1}^{\infty}\frac{2}{\lambda_n^3J_1(\lambda_n)}\left(\frac{\beta_2(1-\beta_1We)e^{-\beta_1t}}{(\beta_1-\beta_2)}-\frac{\beta_1(1-\beta_2We)e^{-\beta_2t}}{(\beta_1-\beta_2)}\right)J_0(\lambda_n r)+\frac{1}{4}\left(1-r^2\right)$$

(36-i)

$$\tau_{11}(t,r)=\left(\sum_{n=1}^{\infty}\frac{(8We\mu_1)^{\frac{1}{2}}}{\lambda_n^2J_1(\lambda_n)}\left(\begin{array}{l}\frac{\beta_2^2(1-\beta_1We)(e^{-2\beta_1t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2(1-2\beta_1We)}+\frac{\beta_1^2(1-\beta_2We)(e^{-2\beta_2t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2(1-2\beta_2We)}\\-\frac{\beta_1\beta_2(2-\beta_1We-\beta_2We)(e^{-(\beta_1+\beta_2)t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)^2(1-\beta_1We-\beta_2We)}\\+\frac{\beta_2(2-\beta_1We)(e^{-\beta_1t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)(1-\beta_1We)}-\frac{\beta_1(2-\beta_2We)(e^{-\beta_2t}-e^{-\frac{1}{We}t})}{(\beta_1-\beta_2)(1-\beta_2We)}\end{array}\right)^{\frac{1}{2}}J_1(\lambda_n r)\right)^2+\frac{We\mu_1}{2}(1-e^{-\frac{1}{We}t})r^2 \tag{36-ii}$$

$$\tau_{12}(t,r)=\sum_{n=1}^{\infty}\frac{-2\mu_1}{\lambda_n^2J_1(\lambda_n)}\left(\frac{\beta_2e^{-\beta_1t}}{(\beta_1-\beta_2)}-\frac{\beta_1e^{-\beta_2t}}{(\beta_1-\beta_2)}\right)J_1(\lambda_n r)-\frac{\mu_1}{2}r \tag{36-iii}$$

As we have $J_0(\lambda_n)=0$, the graph of this equation is given as;

5 CONCLUSION

In this research paper, investigation of analytical and numerical solutions viscoelastic fluid flow without porous media in circular pipes was presented. The main purpose of this research paper was to make mathematical models and to obtain the analytical solutions of the problems arising in the analysis of viscoelastic fluid flow in pipes without porous media medium coupled with constant viscosity Oldroyd–B Constitutive Model. Symmetry method is used to find the exact solutions of the problem and most imperative research of this paper. In this paper, firstly we obtained the steady state solution for non homogenous PDE's and changed these equations in new dependent variable which is homogenous subject to same boundary conditions and new initial conditions. Lie-point symmetries have been calculated of the new dependent of PDE's system and to find an invariance group of the problem. We used the symmetries and tried to reduce the PDEs system into ODE's system and then solved it analytically, so obtained the invariants solutions of the problem. Also, the numerical results of the problem of PDE's system are achieved taking on computer by ND–Solver in Mathematica which were compared with analytical solutions, then they were establish to be in excellent deal with the analytical results

REFERNCES:

Al-Fariss, T. and K. L. Pinder, (2006),‘ ‘Flow through Porous media of a shear-thinning liquid with yield stress’, Can. J. Chem. Eng, 65, 391-404.

Arada N. and A. Sequeira (2003) ‘Strong Steady Solutions for a Generalized Oldroyd-B Model with Shear Dependent Viscosity in a Bounded Domain’, Mathematical Models in Applied Sciences, World Scientific Publishing Company, Vol. 13, No. 9,

Abdel-Malek, M. B., N. A. Badran and H. S. Hassan, (2002), Solution of the Rayleigh problem for a power law non-Newtonian conducting fluid via group method, Int. J. Eng. Sci. 40 1599-1609.

Basov, S., (2004). “Lie Groups of PDE’s and Their Application to the Multidimensional Screening Problems” , Department of Economics, the University of Melbourne.

Bird, R. B., G. C. Dai, and B. J. Yarusso, (2003),‘ ‘The rheology and flow of viscoplastic materials’ Rev. Chem. Eng.**,** Vol.1, 1-70.

Bird, R.B., and O. Hassager, (2001),‘ ‘Armstrong, R C, Dynamics of polymeric liquids’, Vol.1: fluid mechanics, 2nd Edition, Wiley, NewYork.

Bluman, G. W. and Kumei, S., (1989), ‘Handbook of Symmetries and Differential Equations’, New York, Springer.

Chen, C. I., T. Hayat and J. L. Chen, (2006), ’Exact solution for the unsteady flow of a Burgers fluid in a duct induced by time dependent prescribed volume flow rate’ , Heat Mass Transfer 43, 85-90

Dai, G. C. and R. B. Bird, (1981), ‘Radial flow of a Bingham fluid between two fixed circular disks’, J. Non-Newtonian Fluid Mech., Vol.8, 349-355.

Fetecau C. and C. Fetecau, (2005), ‘Unsteady flow of Oldroyd-B fluid in a channel of rectangular cross-section’ Int. J. Non-Linear Mech. 40, 1214 -1219

Ibragimov, N. H., (1999). Handbook of Elementary Lie Groups Analysis and Ordinary Differential Equation, New York, Wiley,

Kwang S. P., and Y. D. Kwon (2009), ‘Numerical description of start-up viscoelastic plane Poiseuille flow’, Korea Australia Rheology Journal, vol. 21, No 1, 47-58.

Moran, M. J. and R. A. Gaggioli, (2012),‘ ‘Reduction of the number of variables in system of Partial Differential Equations with auxiliary conditions, SLAM. J. Appl. Math. 202-215

Oldroyd, J. G. (1958) ‘Non-Newtonian effects in steady motion of some idealized elastico–viscous liquids, Proc. Royal Soc. A245 278–297.

Olver, P. J. (1986), ‘Applications of Lie groups to Differential Equations, Springer, New York.

Phan-Thien, N., and R. I. Tanner, (1997), ‘A new constitutive equation derived from network theory’, J.Non-Newtonian Fluid Mech.2 353–365.

Rajagopal K. R. and A. S. Gupta, (2010),‘ ‘An exact solution for flow of a non-Newtonian fluid past an infinite plate’, Meccanica 19, 158-160.

Rajagopal K. R. and T. Y. Na, (2001),‘Natural convection flow of non-Newtonian fluid between two vertical plates’, Acta Mech. 58, 239-240.

Owens; R. G. and T. N. Phillips (2002), ‘Computational Rheology’ Imperial College Press, vol. 29, 91-116.

Os;van R.G.M andT. N. Phillips. (2004), ‘Spectral element methods for transient viscoelastic flow problems. Journal of Computational Physics’, 201 (1) 286 – 314.

Taha S., (2010). ‘Flow of non-Newtonian Fluid in Porous Media’, Journal of Polymer Science Part B polymer Physics

Taha S., (2009), ‘Modeling the Flow of a Bautista-Manero Fluid in Porous Media’, 121, 12, 25- 31, 32-33

Tan W. C. and T. Masuoka, (2006) Stokes’ Velocity overshoot of start-up flow for a Maxwell fluid in a porous half space’, Chinese Physics, vol. 15, No. 11,. 2644-2650,

Tan W.C and T. Masuoka (2007) ‘Stability analysis of a Maxwell fluid in a porous medium heated from below’, Physics Letters A, Vol. 360, No.3, 454-460

Vafai, K., (2002), ‘Handbook of Porous Media’, Marcel Dekker, New York

Wafo S. C. (2005), ‘Invariant solutions of the unidirectional flow of an electrically charged power law non-Newtonian fluid over a flat plate in presence of a transverse magnetic field’, Comm. in Nonlinear Science and Numerical Simulation, 10, 537-54.