SindhUniv. Res. Jour. (Sci. Ser.) Vol. 48 (2) 367-370 (2016)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

A computational method for fourth order partial integro-differential equation usingseptic B-spline

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Received 18th June 2015 and Revised 29th January 2016

Abstract: A collocation method using septic B-spline functions is presented for approximate solution of a fourth order partialintegro-differential equation with a weakly singular kernel. The second order time derivative is descritized by backward difference formula whereas the fourth order spatial derivativeis approximated usingseptic B-spline functions. The scheme is tested through a benchmark problem. Accuracy of the present scheme is assessed viaL_∞ and L_2 error norms. The scheme is examined using different number of collocation points, time step sizes and time levels. Comparison with quintic B-splinecollocation method is also made to validate the proposed method.

Keywords: Collocation method, Septic B-Spline functions, Partial integro-differential equation, Weakly singular kernel.

1. INTRODUCTION

Partial integro-differential equations (PIDEs) have key role in the real world applications. Some applications of these equations are observed in the modeling of heat flow in materials with memory (Gurtin and Pipkin, 1968, Miller, 1978), viscoelasticity and viscoelastic mechanics (Christensen, 1971, Lodge *et al.*, 1985, Renardy, 1989) in the modeling of thin beams and plates, strain gradient elasticity and phase separation in binary mixtures (Engel *et al.* 2002). Several authors have obtained the solutionof different types of PIDEs including fourth order PIDEs through various methods (Yanik and Fairweather, 1988, Lin and Zhang, 1991, Tang, 1993, Dehghan 2004, Yoon *et al.* 2012, Yang *et al.* 2013, Zhang *et al.* 2013, Ali *et al.* 2015, Ahmad *et al.* 2015, Ali *et al.*2016). In the next section septic B-spline collocation method is constructed for the numerical solution of a fourth order PIDE with a weakly singular kernel (Zhang *et al.* 2013). Some important features of septic B-spline can be found in (Haq *et al*., 2013).

2. CONSTRUCTIONOFTHESEPTIC B-SPLINE COLLOCATION METHOD

The fourth order PIDE(Zhang *et al.* 2013) is

$$\frac{\partial^2 v(\xi,\tau)}{\partial \tau^2}+\int_0^\tau (\tau-\sigma)^{-\alpha} v_{\xi\xi\xi\xi}(\xi,\sigma)d\sigma = f(\xi,\tau),$$

$\xi \in [a,b],\ \tau > 0$,(1)

subject to the initial condition

$$\left.\begin{aligned} v(\xi,0)&=\varphi_1(\xi)\\ v_\tau(\xi,0)&=\varphi_2(\xi)\end{aligned}\right\},\ \xi\in[a,b], \qquad (2)$$

and the boundary conditions are as follows:

$$\left.\begin{aligned} v_\xi(a,\tau)=g_0(\tau), v_\xi(b,\tau)=g_1(\tau), \tau\geq 0,\\ v_{\xi\xi}(a,\tau)=h_0(\tau), v_{\xi\xi}(b,\tau)=h_1(\tau), \tau\geq 0,\\ v_{\xi\xi\xi}(a,\tau)=p_0(\tau), v_{\xi\xi\xi}(b,\tau)=p_1(\tau), \tau\geq 0,\end{aligned}\right\} (3)$$

where$0\leq\alpha\leq 1$.

The space interval $[a,b]$ is descritized into Ncomponents of equal length h by the nodesξ_i, $i=0,1,2,\cdots,N$ such that $a=\xi_0<\xi_1<\cdots<\xi_{N-1}<\xi_N=b$and $h=\frac{b-a}{N}$. The septic B-spline functions $\beta_i(\xi)$, $i=-3,-1,0,1,2,\cdots,N+3$at these knots are defined as

$$\beta_i(\xi)=\frac{1}{h^7}\begin{cases} a_1=(\xi-\xi_{i-4})^7, \\ \xi\in[\xi_{i-4},\xi_{i-3}], \\ a_2=a_1-8(\xi-\xi_{i-3})^7, \\ \xi\in[\xi_{i-3},\xi_{i-2}], \\ a_3=a_2+28(\xi-\xi_{i-2})^7, \\ \xi\in[\xi_{i-2},\xi_{i-1}], \\ a_3=a_3-56(\xi-\xi_{i-1})^7, \\ \xi\in[\xi_{i-1},\xi_i], \\ (\xi_{i+4}-\xi)^7-8(\xi_{i+3}-\xi)^7 \\ +28(\xi_{i+2}-\xi)^7-56(\xi_{i+1}-\xi)^7, \\ \xi\in[\xi_i,\xi_{i+1}], \\ (\xi_{i+4}-\xi)^7-8(\xi_{i+3}-\xi)^7 \\ +28(\xi_{i+2}-\xi)^7,\ \xi\in[\xi_{i+1},\xi_{i+2}], \\ (\xi_{i+4}-\xi)^7-8(\xi_{i+3}-\xi)^7, \\ \xi\in[\xi_{i+2},\xi_{i+3}], \\ (\xi_{i+4}-\xi)^7, \xi\in[\xi_{i+3},\xi_{i+4}], \\ 0, \quad \text{otherwise.}\end{cases} \qquad (4)$$

The set of septic B-splines $\{\beta_{-3},\beta_{-2},\beta_{-1},\beta_0,\beta_1,\beta_2,\cdots,\beta_{N+3}\}$ forms a basis for the functions $v(\xi,\tau)$over $[a,b]$. An approximation $V_N(\xi,\tau)$

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to the exact solution $v(\xi,\tau)$ takes the form $V_N(\xi,\tau)=\sum_{i=-3}^{N+3}\gamma_i(\tau)\beta_i(\xi)$,(5)
where$\gamma_i(\tau)$ are unknown time dependent quantities which are to be determined. The nodal values V,V_i',V_i'',V_i''' and V_i'''' at the grid points $\xi_i, i=0,1,2,\dots,N$ areobtained from Eqs. (4)-(5) in the following form

$$\left.\begin{aligned} V_i &= \gamma_{i-3}+120\gamma_{i-2}+1191\gamma_{i-1}+2416\gamma_i\\ &\quad +1191\gamma_{i+1}+120\gamma_{i+2}+\gamma_{i+3},\\ V_i' &= \tfrac{7}{h}(-\gamma_{i-3}-56\gamma_{i-2}-245\gamma_{i-1}\\ &\quad +245\gamma_{i+1}+56\gamma_{i+2}+\gamma_{i+3}),\\ V_i'' &= \tfrac{42}{h^2}(\gamma_{i-3}+24\gamma_{i-2}+15\gamma_{i-1}-80\gamma_i\\ &\quad +15\gamma_{i+1}+24\gamma_{i+2}+\gamma_{i+3}),\\ V_i''' &= \tfrac{210}{h^3}(-\gamma_{i-3}-8\gamma_{i-2}+19\gamma_{i-1}\\ &\quad -\gamma_{i+1}+8\gamma_{i+2}+\gamma_{i+3}),\\ V_i'''' &= \tfrac{840}{h^4}(\gamma_{i-3}-9\gamma_{i-2}+16\gamma_i\\ &\quad -9\gamma_{i+1}+\gamma_{i+3}), \end{aligned}\right\}(6)$$

where dashes represent derivatives with respect to the space variableξ and $V_i=v(\xi_i,\tau)$. The time level is denoted by $\tau_n=n\Delta\tau$, where $\Delta\tau$ is the increment in time. Approximating the problem at $\tau=\tau_{n+1}$by discretizing the time derivative $v_{\tau\tau}(\xi,\tau_{n+1})$ in Eq. (1) using backward differences

$$v_{\tau\tau}(\xi,\tau_{n+1})\approx\frac{v^{n+1}(\xi)-2v^n(\xi)+v^{n-1}(\xi)}{\Delta\tau^2},\quad \xi\in[a,b],\ n\geq 1,$$

to obtain

$$\frac{v^{n+1}(\xi)-2v^n(\xi)+v^{n-1}(\xi)}{\Delta\tau^2}+\int_0^{\tau_{n+1}}(\tau_{n+1}-\sigma)^{-\alpha}v_{\xi\xi\xi\xi}(\xi,\sigma)=f(\xi,\tau_{n+1}).\ (7)$$

The integral part in Eq. (7) is evaluated as (Zhang *et al.*, 2013, Ahmad *et al*. 2015):

$$\begin{aligned} &=\int_0^{\tau^{n+1}}\sigma^{-\alpha}v_{\xi\xi\xi\xi}(\xi,\tau_{n+1}-\sigma)d\sigma,\\ &=\sum_{m=0}^{n}\int_{\tau_m}^{\tau_{m+1}}\sigma^{-\alpha}v_{\xi\xi\xi\xi}(\xi,\tau_{n+1}-\sigma)d\sigma,\\ &\approx\sum_{m=0}^{n}\sigma^{-\alpha}v_{\xi\xi\xi\xi}(\xi,\tau_{m-n+1})\int_{\tau_m}^{\tau_{m+1}}\sigma^{-\alpha}\,d\sigma,\\ &=\tfrac{\Delta\tau^{1-\alpha}}{1-\alpha}\textstyle\sum_{m=0}^{n}v_{\xi\xi\xi\xi}(\xi,\tau_{m-n+1})((m+1)^{1-\alpha}-m^{1-\alpha}). \end{aligned}\quad(8)$$

Substituting Eq. (8) in Eq. (7) and rearranging, we obtain

$$v^{n+1}(\xi)+\tfrac{\Delta\tau^{3-\alpha}}{1-\alpha}b_0v_{\xi\xi\xi\xi}^{n+1}=2v^n(\xi)-v^{n-1}(\xi)+\tfrac{\Delta\tau^{3-\alpha}}{1-\alpha}\textstyle\sum_{m=1}^{n}b_mv_{\xi\xi\xi\xi}^{n-m+1}(\xi,\tau_{n-m+1})+\Delta\tau^2f^{n+1}(\xi).\ (9)$$

To develop the septic B-spline scheme, use Eq. (6) at $\xi=\xi_i,\ i=0,1,2,\dots,N$and substituting theapproximate values of $v^{n+1}(\xi)$ and $v_{\xi\xi\xi\xi}{}^{n+1}(\xi)$ in Eq. (9), we getthe following $(N+1)\times(N+6)$linear system:

$$d_1\gamma_{i-3}^{n+1}+d_2\gamma_{i-2}^{n+1}+d_3\gamma_{i-1}^{n+1}+d_4\gamma_i^{n+1}+d_5\gamma_{i+1}^{n+1}+d_6\gamma_{i+2}^{n+1}+d_7\gamma_{i+3}^{n+1}=F_i,(10)$$

where$d_1=1+q, d_2=120, d_3=1191-9q, d_4=2416+16q, d_5=1191-9q, d_6=120, d_7=1+q$, $q=\frac{840}{h^4}\frac{\Delta\tau^{3-\alpha}}{1-\alpha}$ and $F_i=2v^n(\xi_i)-v^{n-1}(\xi_i)+\frac{\Delta\tau^{3-\alpha}}{1-\alpha}\sum_{j=1}^{n}b_jv_{\xi\xi\xi\xi}^{n-j+1}(\xi_i)+\Delta\tau^2f^{n+1}(\xi_i)$.

By eliminating the parameters $\gamma_{-1},\gamma_{-2},\gamma_{-3},\gamma_{N+1},\gamma_{N+2}$ and γ_{N+3} using the following relations obtained from Eq. (3) and Eq. (10), we get a system of $N+1$ linear equations in $N+1$ unknowns:

$$\begin{aligned} \gamma_{-1} &= -\frac{40}{27}\gamma_0+\frac{14}{9}\gamma_1+\frac{8}{9}\gamma_2+\frac{1}{27}\gamma_3-\frac{1}{2268}h_0h^2\\ &\quad -\frac{1}{1134}g_0h-\frac{1}{17010}p_0h^3,\\ \gamma_{-2} &= -\frac{220}{27}\gamma_0-\frac{15}{18}\gamma_1-\frac{35}{9}\gamma_2-\frac{11}{54}\gamma_3+\frac{11}{4536}h_0h^2\\ &\quad +\frac{17}{9072}g_0h+\frac{23}{54432}p_0h^3,\\ \gamma_{-1} &= -\frac{280}{3}\gamma_0+35\gamma_1+56\gamma_2+\frac{10}{3}\gamma_3-\frac{1}{36}h_0h^2\\ &\quad -\frac{2}{63}g_0h-\frac{1}{108}p_0h^3,\\ \gamma_{N+1} &= \tfrac{1}{27}\gamma_{N-3}+\tfrac{8}{9}\gamma_{N-2}+\tfrac{14}{9}\gamma_{N-1}-\tfrac{40}{27}\gamma_N-\tfrac{1}{2268}h_1h^2\\ &\quad -\tfrac{1}{1134}g_1h-\tfrac{1}{17010}p_1h^3, \end{aligned}$$

$$\begin{aligned} \gamma_{N+2} &= -\frac{11}{54}\gamma_{N-3}-\frac{35}{9}\gamma_{N-2}-\frac{55}{18}\gamma_{N-1}+\frac{220}{27}\gamma_N\\ &\quad +\frac{11}{4536}h_1h^2+\frac{17}{9072}g_1h+\frac{23}{54432}p_1h^3,\\ \gamma_{N+3} &= \frac{10}{3}\gamma_{N-3}+56\gamma_{N-2}+35\gamma_{N-1}-\frac{280}{3}\gamma_N-\frac{1}{36}h_1h^2\\ &\quad -\frac{2}{63}g_1h-\frac{1}{108}p_1h^3. \end{aligned}$$

This system can be solved by a septa-diagonal solver forγ_i^n, $i=0,1,2,3\cdots N$. Finally the approximate solution $V(\xi,\tau)$ is evaluated using Eq. (5).

3. NUMERICAL TEST

The present method given in (10) for the solution of the problem defined by (1)-(3) is tested through the following example and its accuracy is assesed via error norms L_2 and L_∞ and by comparison of the results with (Zhang *et al*., 2013).**Example:** Consider Eqs. (1)-(3) for $\alpha=\frac{1}{2}$. $f(\xi,\tau)$is chosen so that the exact solution of the problem (Zhang *et al*.) is given by
$v(\xi,\tau)=2(\tau^2+2\tau+1)\sin^2\pi\xi$.
Initial and boundary conditions are obtained from the exact solution as follows:

$$\left.\begin{aligned} v_\xi(0,\tau)=0,\ v_\xi(1,\tau)=0,\\ v_{\xi\xi}(0,\tau)=4\pi^2(\tau^2+\tau+1),\\ v_{\xi\xi}(1,\tau)=4\pi^2(\tau^2+\tau+1),\\ v_{\xi\xi\xi}(0,\tau)=0,\ v_{\xi\xi\xi}(1,\tau)=0. \end{aligned}\right\}$$

In **(Table-1)**, the computed results are recorded which are obtained using$N=10$,$M=10$and different values of $\Delta\tau$. In **(Table 2)**, the calculated results are shown for the parameters $N=10,\Delta\tau=10^{-6}$and different values of M.In **(Table 3)**, the computed results are provided for the parameters $M=10,\Delta\tau=10^{-6}$and different values of N. **(Fig. 1)**, shows exact and numerical solutions for$M=50,N=100$and $\Delta\tau=10^{-6}$. In **(Fig. 2)**, error in the solution is shown over the space interval [0, 1] using $M=50,N=100$ and $\Delta\tau=10^{-6}$ while **(Fig. 3)**, represents solutionup to$M$=50.The results are compared with quintic method (Zhang *et al.*, 2013).

Table 1:Error norms using$N=10,\ M=10,\ \alpha=\frac{1}{2}$

	Septic B-spline		(Zhang *et al.*, 2013)	
$\Delta\tau$	L_∞	L_2	L_∞	L_2
10^{-4}	3.53×10^{-5}	7.19×10^{-6}	1.48×10^{-4}	2.36×10^{-4}
10^{-5}	1.11×10^{-7}	2.49×10^{-8}	4.68×10^{-7}	7.49×10^{-7}
10^{-6}	3.48×10^{-10}	1.00×10^{-10}	1.47×10^{-9}	2.37×10^{-9}
10^{-7}	1.05×10^{-12}	5.33×10^{-13}	4.66×10^{-12}	7.50×10^{-12}

Table 2: Error norms using$N=20,\ \alpha=\frac{1}{2},\ \Delta\tau=10^{-6}$

	Septic B-spline		(Zhang et al., 2013)	
M	L_∞	L_2	L_∞	L_2
10	3.48×10^{-10}	1.00×10^{-10}	1.47×10^{-9}	2.37×10^{-9}
20	1.17×10^{-9}	2.93×10^{-10}	8.52×10^{-9}	1.36×10^{-8}
30	2.31×10^{-9}	5.48×10^{-10}	2.36×10^{-8}	3.78×10^{-8}
40	3.72×10^{-9}	8.54×10^{-10}	4.85×10^{-8}	7.77×10^{-8}
50	5.34×10^{-9}	1.20×10^{-9}	8.49×10^{-8}	1.35×10^{-7}

Table 3: Error norms using $M=20,\ \alpha=\frac{1}{2},\Delta\tau=10^{-6}$

	Septic B-spline	
N	L_∞	L_2
10	1.32×10^{-9}	4.42×10^{-10}
20	4.98×10^{-9}	8.28×10^{-10}
50	2.92×10^{-8}	2.71×10^{-9}
100	1.15×10^{-7}	7.32×10^{-9}

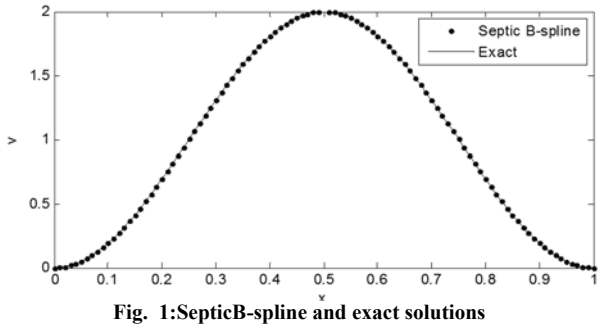


Fig. 1:SepticB-spline and exact solutions

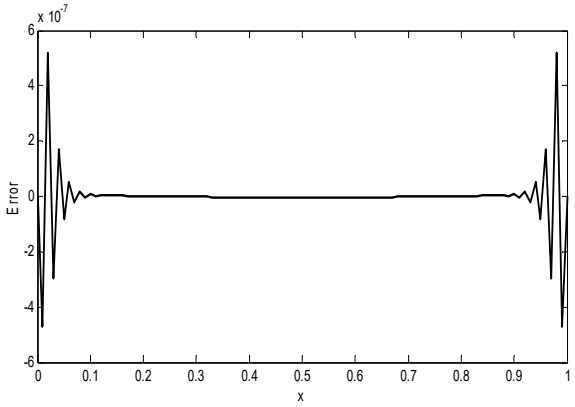


Fig. 2: Error graph

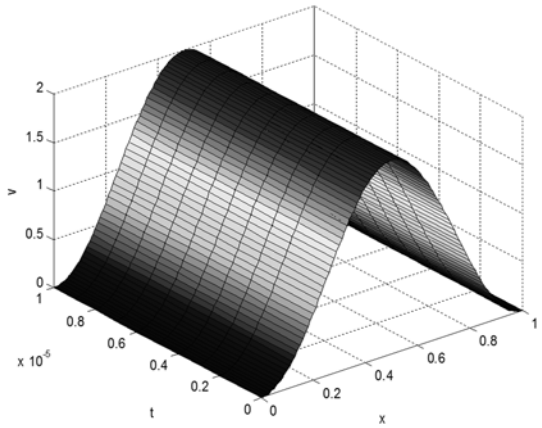


Fig. 3: Septic B-spline solutions up to $M=50$

4. CONCLUSION

In this article, we developed collocation scheme using septic B-spline to the approximate solution of a fourth order partial integro-differential equation with a weakly singular kernel. The proposed method provides better accuracy and numerical simulation has been shown a good agreement with exact solution. Thus the method is economical and simple to implement for solution ofthe fourth order partial integro-differential equation with a weakly singular kernel. The results are also compared with quantic B-spline method from literature. However, to obtain a higher accuracy a small time step is required in the scheme.

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