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Existence and Uniqueness of Positive Solution to Boundary Value Problem of Fractional Differential Equations

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Abstract: This article is concerned with the study of positive solution to a class of nonlinear fractional differential equation with boundary conditions. We use some important theorems from fixed point theory to obtain the required results. At the end we apply these results on example to show its applicability.

Keywords: Fractional differential order equation; Fixed point theorem; Boundary value problem; Positives solution; Green's function

1. INTRODUCTION

In the past few years' fractional calculus have fascinated scientists by virtue of its tremendous applications in many branches of sciences e.g. population dynamics, finance, physics etc. (1993), (2006). During the last decades, with the development of differential and integral equations as well as difference equations, a lot of fractional differential equations have been expounded. (2000), (2001) etc. These handy discoveries play an important role in the research of boundedness of solution, global existence, stability of solutions of differential and integral equations as well as difference equations.

Also for further theory and applications of fractional differential equations, the readers who are engrossed in details of the subject, we referee to see the monographs Hilfer (2000) and Podlubny (1999). The distinguish features of fractional differential equations in that it out line memory and transmitted properties of numerous mathematical models. As compared with other differential equation the fractional order models are more realist and workable as compared to the classical integral order models. Fractional differential equations have various advantages in many engineering, physical and numerical science as a new way of modeling dynamical system and other observable occurrence in fields of fluid mechanics, physical chemistry, biomathematics, and so forth, involves derivatives of fractional order. In conclusion the subject of fractional differential equations is gaining much importance and attention, (Laskin (2000), El-Saka (2014), El-Sayed and Gaafar (2001), the references therein). The qualitative theory of existence of positive solutions to integer order differential equations were well studied as see Liu(2008) and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations has been given less attention and still in the initial stages and many aspects of this theory need to be explored. The area dedicated to the existence theory of positive solutions to fractional order boundary value problems and their coupled system were well studied in last few years, for detail see Benchohra (2008), Ahmad (2009), Shah and Khan (2015) and (2016) and Khan and Shah (2015) and Z. Bai and H. Lu(2005). In all these articles the concerned studies was carried out by using standard technique of functional analysis and fixed point theory.

The present article will investigate the existence of positive solution to the following problem of non-linear differential equations of frictional order given by

$$ {}^{c}D^{P}w(t) = h\left(t, w(t), {}^{c}D^{p-1}w(t)\right), \quad 1 < p \leq 2, $$
$$ t \in J = [0,1] $$
$$ w(0) = \xi w(1), \; {}^{c}D^{\beta}u(0) = \xi\, {}^{c}D^{\beta}w(1), \quad 0 < \beta < 1, \; \xi \in (0,1) \qquad (1) $$

and $h: J \times R \times R \to R$ is given nonlinear continuous function, where ${}^{c}D^{\propto}_{+0}$ represents Caputo fractional order derivative. Since it have been observed in the literature that usually fixed-point theory is used as a main tool for the existence and uniqueness of solution. Therefore, we will also use the fixed-point theorem of Laray Shouder type as well as Banach Fixed point theorem to exhibit necessary and sufficient conditions for the existence of positive solution as well as the uniqueness. We will elaborate the main results by providing an example.

2. MATERIALS AND METHODS

As it is known that expounding new results we need some basic and preliminaries results, which are used as main tools. In this section, we preset some basic definition and results, necessary lemmas that will be used

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in the remainder of this article. The concerned results are recalled from Bai and Lu (2005), Khan and Shah (2015), Podlubny (1999) and Agarwal (2004). We initially evolve the associated Greens function for our new model of BVP. Subsequently the problem (1) will be reduced to corresponding Fredhom integral equation of 2nd-kind. Finally, with the help of preliminaries results and fixed point theorem we will exhibit, the existence as well as uniqueness of the solution.

Definition 2.1. The well known, Riemann- Liouvili fractional integral of order "$p > 0$" of the function $y:(0,\infty) \to R$, where "y" is continuous function is defined as

$$I^p\big(y(t)\big) = \frac{1}{\Gamma p}\int_0^t (t-s)^{p-1} y(s)ds.$$

Provided that the integral exists.

Definition 2.2: For a function $y:(0,\infty) \to R$, the Riemann-Liouville derivative of fractional order "$p > 0$" is defined by

$$D^p[y(t)] = \frac{1}{\Gamma(n-p)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-p-1} y(s)ds,$$

provided that integral exists , where $n = [p] + 1$, $[p]$ is integral part of the real value p.

Definition 2.3 Assume that $y:(0,\infty) \to R$ be a continuous function, then the Caputo's derivative of "y" of order "p" is defined by,

$${}^c D^p[y(t)] = \frac{1}{\Gamma(n-p)}\int_0^t (t-s)^{n-p-1}\frac{d^n y(s)}{dt^n}ds\ .$$

Where $n = [p] + 1$, and $[p]$ is integral part of $p \in R$.

Theorem 2.4. Let $n-1 < p < n$, then the differential equations of arbitrary order p given below

${}^c D^p y(t) = 0$, has solutions
$y(t) = \sum_{k=0}^{n-1} C_k t^k$, where $C_k \in R, k = 0,1,2, ..., n = [p] + 1$.

Theorem 2.5. : Let $n-1 < p < n$, then the following differential equation has a solution

$I^p[{}^c D^{p-1}]y(t)) = y(t) + \sum_{i=0}^{n-1} C_i t^i$,
Where, $C_i \in R, i = 0,1,2, and\ \ n = [p] + 1$.

Theorem 2.6. If $h: I \times R \times R \to R$ is function such that $f \in C[a,b]$ and $1 < p < 2$, then

$$D^p y(t) = f\left(t, u(t), {}^c D^{p-1} y(t)\right).$$

Along with the subsidiary conditions

$$u(0) = \boldsymbol{\xi} u(1),\ {}^c D^\beta u(0) = \xi\, {}^c D^\beta u(1),,$$

Where $0 < \xi < 1, 0 < t < 1$,
has a unique solution given by

$$w(t) = \int_0^1 G(t,s) f\left(s, w(s), {}^c D^{p-1} w(s)\right) ds.$$

Where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \frac{(1-\xi)(t-s)^{p-1} + \xi(1-s)^{p-1}}{\Gamma p(1-\xi)} - \frac{\Gamma(2-\beta)(\xi+(1-\xi)t)(1-s)^{p-\beta-1}}{\Gamma(p-\beta)(1-\xi)}, \\ 0 \le s \le t \le 1, \\ \frac{\xi(1-s)^{p-1}}{\Gamma\propto(1-\xi)} - \frac{\Gamma(2-\beta)(\xi+(1-\xi)t)(1-s)(1-s)^{p-\beta-1}}{\Gamma(\alpha-\beta)(1-\xi)}, \\ 0 \le t \le s \le 1 \end{cases} \quad (2)$$

Solution: Let $y \in C([0,1], R)$, then solution of ${}^c D^p w(t) = y(t)$ subject to the boundary conditions $w(0) = \boldsymbol{\xi} w(1),\ {}^c D^\beta w(0) = \xi\, {}^c D^\beta w(1)$, $0 < \xi < 1, 0 < t < 1$ is obtain as follow:
In view of Theorem 2.5 and Definition (2.1), we can write (1) as

$I^p({}^c D^p y(t)) = I^p\, y(t)$
$\Rightarrow y(t) = C_0 + C_1 t + I^p y(t)$. (3)
Now using boundary condition $w(0) = \xi w(1)$, $D^\beta u(0) = \xi D^\beta w(1)$, in (3),
we proceed as
$C_0 = \xi C_0 + \xi C_1 + \xi I^p y(1) \Rightarrow (1-\xi)C_0 = \xi C_1 + \xi I^p y(1)$, (4)

Now ${}^c D^\beta w(t) = C_1 \frac{t^{1-\beta}}{\Gamma(2-\beta)} + I^{p-\beta} y(t)$

${}^c D^\beta w(0) = {}^c D^\beta\ w(1) = \xi\left[\frac{C_1}{\Gamma(2-\beta)} + I^{p-\beta} y(1)\right]$
$\Rightarrow\ \ C_1 = -\Gamma(2-\beta) I^{p-\beta} y(1)$,
putting in (4)
$(1-\xi)C_0 = -\xi\Gamma(2-\beta) I^{p-\beta} y(1) + \xi I^p y(1)$
$C_0 = \frac{\boldsymbol{\xi} I^p y(1) - \boldsymbol{\xi}\Gamma(2-\beta) I^{p-\beta} y(1)}{(1-\boldsymbol{\xi})}$, put in (3)
$w(t) = \xi\left[\frac{I^p y(1) - \Gamma(2-\beta) I^{p-\beta} y(1)}{1-\xi}\right] - t\Gamma(2-\beta) I^{p-\beta} y(1) + I^p y(1)$. (5)
Which implies that $u(t) = \int_0^1 G(t,s) y(s) ds$. (6)
Where $G(t,s)$ define the following properties:
If $p - \beta < 1$ the green function $G(t,s)$ become unbounded but the function
$t: \to \int_0^1 G(t,s) y(s) ds$ is continuous on J, so attain its supermom value say G^*
$G^* = \sup_(t \in J) \int_0^1 |G(t,s)| ds.$

3. RESULTS AND DISCUSSION

In the previous section we recalled some basic and preliminary results to be used in the existence theory of the solution. The present section of the paper is devoted to discuss uniqueness and the existence of at least one positive solution to the boundary value problem (1).

Theorem 3.1: Assume that

(A_0) $h: J \times R \times R \to R$ is continuous;

(A_1) We can find a constant $k > 0$ so that $t \in J$ and for all $w, v, \overline{w}, \bar{v} \in R$,

$|h(t, w, v,) - h(t, \overline{w}, \bar{v})| \leq k|(w - \overline{w}) + (v - \bar{v})|$.

If $\max\{2G^*k, 2k \frac{\Gamma(3-p)\Gamma(p-\beta+1)+\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\} < 1$, then boundary value problem (1) has a unique solution.

Proof: In order to prove the existence of solution of the problem (1), we will an operator and by fixed-point theorem will show that this operator have a solution, which will be the solution of required BVP. Assume an operator $T: \bar{C}(J, R) \to \bar{C}(J, R)$ defined by

$$Tw(t) = \int_0^1 G(t,s)h\left(s, w(s), {}^cD^{p-1}w(s)\right)ds \qquad (7)$$

is continuous mapping with fixed $u(t)$, where $w(t)$ is the required solution of boundary value problem (1). Now by lemma 2.4, we have from (5),

$${}^cD^{p-1}w(t) = Iy(t) - \Gamma(2-\beta)\frac{t^{2-p}}{\Gamma(3-p)}I^{p-\beta}y(1)$$

$$= \int_0^t h\left(s, w(s), {}^cD^{p-1}w(t)\right)ds - \frac{\Gamma(2-\beta)t^{2-p}}{\Gamma(3-p)\Gamma(p-\beta)}\int_0^t (1-S)^{p-\beta-1}h\left(s, w(s), {}^cD^{p-1}w(t)\right)ds .$$

Clearly by continuity of f, both $Tw(t)$ and ${}^cD^{p-1}Tw(t)$ are continuous on J. Also let$w, \bar{w} \in \bar{C}(J, R)$ then for each $t \in J$

$|Tw(t) - T\bar{w}(t)|$

$$\leq \sup \in jG(t,s)\int_0^1 \left|h\left(s, w(s), {}^cD^{p-1}w(s)\right) - h\left(s, \bar{w}(s), {}^cD^{p-1}\bar{w}(s)\right)\right| ds$$

$$\leq G^*k\left\{|w - \bar{w}| + \left|{}^cD^{p-1}w(s) - {}^cD^{p-1}\bar{w}(s)\right|\right\}$$

$$\leq G^*k max\left\{\left||w - \bar{w}|\right|_\infty, \left|\left|{}^cD^{p-1}w - D^{p-1}\bar{w}\right|\right|_\infty\right\} + G^*k max\left\{\begin{matrix}\left||w - \bar{w}|\right|, \\ \left|\left|{}^cD^{p-1}w - {}^cD^{p-1}\bar{w}\right|\right|\end{matrix}\right\},$$

Which implies that

$$\left||Tw(t) - T\bar{w}(t)|\right|_\infty \leq 2G^*k\left||w - w|\right|_c . \qquad (8)$$

and

$$\left|{}^cD^{p-1}Tw(t) - {}^cD^{p-1}T\bar{w}(t)\right|$$

$$\leq \int_0^1 |h\left(s, w(s), {}^cD^{p-1}w(s)\right) - h\left(s, \bar{w}(s), {}^cD^{p-1}\bar{w}(s)\right)| ds$$

$$+\frac{\Gamma(2-\beta)t^{2-p}}{\Gamma(3-p)\Gamma(p-\beta)}\int_0^1 (1-S)^{p-\beta-1}\left|h\left(s, w(s), {}^cD^{p-1}w(s)\right) - h\left(s, \bar{w}(s), {}^cD^{p-1}\bar{w}(s)\right)\right| ds.$$

$$\leq k\left\{|w - \bar{w}| + \left|{}^cD^{p-1}w - {}^cD^{p-1}\bar{w}\right|\right\}t + \frac{\Gamma(2-\beta)t^{2-p}}{\Gamma(3-p)\Gamma(p-\beta)}\frac{(1-s)^{p-\beta}}{p-\beta}|_0^t(|w - \bar{w}|$$

$$+|{}^cD^{p-1}w(s) - {}^cD^{p-1}\overline{w}(s)|).$$

Hence $\left||{}^cD^{p-1}Tw(t) - {}^cD^{p-1}T\bar{w}(t)|\right|_\infty \leq 2k\left||u - \bar{u}|\right|_c + \frac{\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\left||w - \bar{w}|\right|_c$

$$\leq \left(2k + \frac{\Gamma(2-\beta).1}{\Gamma(3-p)\Gamma(p-\beta+1)}\right)\left||w - \bar{w}|\right|_c. \qquad (9)$$

From (8) and (9), we have

$$||Tw(t) - Tw(t)|| \leq max\left(2G^*k, 2k + \frac{\Gamma(3-p)\Gamma(p-\beta+1)+\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\right)\left||w - \bar{w}|\right|_c. \qquad (10)$$

But

$$\max\left\{2G^*k, 2k\left(\frac{\Gamma(3-p)\Gamma(p-\beta+)+\Gamma(2-\beta).1}{\Gamma(3-p)\Gamma(p-\beta+1)}\right)\right\} = d < 1.$$

Which implies that $\left||Tw - T\bar{w}|\right|_c \leq d\left||w - \bar{w}|\right|_c$. Therefore, the operator T is contraction mapping .Hence, by Banach fixed point theorem the BVP (1) has a unique solution.

Theorem3.2: Assume that

(A_3) There exists $q \in C(J, R^+)$ and $\psi: [0,\infty] \to (0,\infty)$ is a continuous and non-decreasing function such that for all $t \in J, w, z \in R$

$$|h(t,w,z) \le q(t)\psi(|z|)|.$$

(A_4) There exists constant $r > 0$ such that

$$r \ge \max\left\{G^* q^* \psi(r), q^* \psi(r) \frac{\Gamma(3-p)\Gamma(p-\beta+)+\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\right\}.$$

Where $q^* = \sup\{q(s), s \in J\}$.

Then boundary value problem (1) has at least one solution on J with $|w(t)| < r$ for each t $\in J$.

Proof: To prove the theorem we will prove that T satisfied all the condition of Shauder fixed point theorem.

Step1: First we will prove that T is continuous. Let us defined a bounded set by $D = \{w \in \tilde{c}(J,R), ||w||_{\tilde{c}} \le r\}$, where r is continuous such that

$$r \ge \max\left\{G^* q\psi(r), q^* \psi(r) \frac{\Gamma(3-p)\Gamma(p-\beta+) + \Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\right\}$$

is closed subset of $\tilde{C}(J,R)$.

If $\{w_n\}$ is convergent sequence in $\tilde{C}(J,R)$ such that $w_n \to w$ as $n \to \infty$. Now $\tilde{C}(J,R)$ is a Banach space so that there exists $\rho > 0$, such that $||w_n||_{\tilde{c}} \le \rho$ then for all $t \in J$,

$$|Tw_n(t) - Tw(t)| \le \int_0^1 |G(t,s)[h\left(s, w_n(s), {}^c D^{p-1} w_n(s) - h\left(s, w(s), {}^c D^{p-1} w(s)\right)\right)| \, ds.$$

Step 2: Let $D \in C(J,R)$ is bounded set, to prove $T(D)$ is also bounded, we will show that $T(D) \subseteq D$, let $w(t) \in D$, then for each $t \in J$, we have

$$|Tw(t)| \le \int_0^t (|G(t,s)||h\left(s, w(s), {}^c D^{p-1} w(s)\right)|) ds$$

$$\le G^* \int_0^t (|h\left(s, w(s), {}^c D^{p-1} w(s)\right)| ds \le G^* q^* \psi(||u||_{\tilde{c}}) \le G^* q^* \psi(r) .$$

Further $\left|{}^c D^{p-1} Tw(t)\right| \le \int_0^t (|G(t,s)||h\left(s, w(s), {}^c D^{p-1} w(s)\right)|) ds$

$$+ \frac{\Gamma(2-\beta)t^{2-p}}{\Gamma(3-p)\Gamma(p-\beta)} \int_0^1 (1-S)^{p-\beta-1} \left|h\left(s, w(s), {}^c D^{p-1} w(s)\right)\right| ds$$

$$\le q^* \psi\left(||w||_{\tilde{c}}\right) \frac{\Gamma(3-p)\Gamma(p-\beta+)+\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}.$$

Hence $||Tw(t)||_{\tilde{c}} \le \max\left\{G^* q^* \psi(r), q^* \psi(r) \frac{\Gamma(3-p)\Gamma(p-\beta+)+\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\right\} \le r.$

Where $r \ge \max\left\{G^* q^* \psi(r), q^* \psi(r) \frac{\Gamma(3-p)\Gamma(p-\beta+)+\Gamma(2-\beta)}{\Gamma(3-p)\Gamma(p-\beta+1)}\right\}$.

Showing that $Tw(t) \in D$, for all $w(t) \in D$, $T(D) \subseteq D$.

Step 3: To prove T maps D into equicontinuous set of $\tilde{C}(J,R)$. Taking $t_1,\ t_2 \in J,$ with $t_1 < t_2$ and $w(t) \in D$, we have $||Tw(t_2) - Tw(t_1)|| \le \int_0^1 (|G(t_2,s) - G(t_1,s)||h\left(s, w(s), {}^c D^{p-1} w(s)\right)|) ds$

$$\le q^* \psi\left(||w||_{\tilde{c}}\right) \int_0^1 (|G(t_2,s) - G(t_1,s)| ds.$$

If $t_2 \to t_1$, we get $||Tw(t_2) - Tw(t_1)|| \to 0,$

and $||T\,{}^c D^{p-1} w(t_2) - T\,{}^c D^{p-1} w(t_1)|| \le |\int_0^{t_2} h\left(s, w(s), {}^c D^{p-1} w(s)\right) - \int_0^{t_1} h\left(s, w(s), {}^c D^{p-1} w(s)\right) ds|$

$$+ \frac{\Gamma(2-\beta)(t^{2-p} - {t_2}^{2-p})}{\Gamma(3-p)\Gamma(p-\beta)} \int_0^1 (1-S)^{p-\beta-1} \left|h\left(s, w(s), {}^c D^{p-1} w(s)\right)\right| ds \le$$

$$q^* \psi\left(max\left(||w||_{\infty}, \left|\left|{}^c D^{p-1} w(t)\right|\right|_{\infty}\right)\right)\left\{t_2 - t_1 + \frac{\Gamma(2-\beta)({t_1}^{2-p} - {t_2}^{2-p})}{\Gamma(3-p)\Gamma(p-\beta)}\right\} \le q^* \psi\left(||w||_{\tilde{c}}\right)\left\{t_2 - t_1 +\right.$$

$\frac{\Gamma(2-\beta)(t_1{}^{2-p}-t_2{}^{2-p})}{\Gamma(3-p)\Gamma(p-\beta)}\Big\}$.

Which implies that $|T\,{}^{c}D^{p-1}w(t_2) - T\,{}^{c}D^{p-1}w(t_1)| \to 0$ as $t_2 \to t_1$.

Thus $\max\left\{\left|\left|Tw(t_2)-Tw(t_1)\right|\right|_{\infty}, \left|\left|T\,{}^{c}D^{p-1}w(t_2) - T\,{}^{c}D^{p-1}w(t_1)\right|\right|_{\infty}\right\} \to 0$

as $t_2 \to t_1$ or $\left|\left|Tw(t_2)-Tw(t_1)\right|\right|_{\tilde{C}} \to 0$ as $t_2 \to t_1$.

Showing that T is completely continuous by Arsela-Ascoli Theorem.

Therefore by Schauder's fixed point theorem T has a fixed point $w(t)$ in D which is the solution of boundary value problem (1) such that $|w(t)| < r$, for all $t \in J$.

4. Example

Consider the fractional boundary value problem

$$ {}^{c}D^{1.5}w(t) = h(t, w(t), {}^{c}D^{0.5}w(t)),\ t \in [0,\ 1], \qquad (11)$$

$$ w(0) = 0.5w(1),\ \ {}^{c}D^{0.5}w(0) = 0.5\,{}^{c}D^{0.5}w(0.5). $$

Here $h(t,w,v) = \frac{1}{25e^{t}+8}\left(\frac{1}{1+2|w|+3|v|}\right) = \dfrac{1}{25e^{t}+8}\left(\dfrac{1}{1+2\,|\,w(t)\,|+3|{}^{c}D^{0.5}w(t)\,|}\right)$

$$|h(t,w,v)-h(t,\bar{w},\bar{v})| \le \frac{1}{33}\left(\frac{1}{1+2|w|+3|v|}-\frac{1}{1+2|\bar{w}|+3|\bar{v}|}\right) \le \frac{1}{33}\left(\frac{1+2|\bar{w}|+3|\bar{v}|-1-2|w|-3|v|}{1+2|w|+3|v|}\right)$$
$$\le \frac{3}{33}(|w-\bar{w}|+|v-\bar{v}|) \le \frac{1}{11}(|w-\bar{w}|+|v-\bar{v}|), \quad \text{for } k=\frac{1}{11}.$$

Which is condition (A_1) with $\left(k=\frac{1}{11}\right)$ and

$$G(t,s) = \begin{cases} \frac{(1-\xi)(t-s)^{p-1}+\xi(1-s)^{p-1}}{\Gamma p(1-\xi)} - \frac{\Gamma(2-\beta)(\xi+(1-\xi)t)(1-s)^{p-\beta-1}}{\Gamma(p-\beta)(1-\xi)},\ 0 \le s \le t < 1 \\ \frac{\xi(1-s)^{p-1}}{\Gamma p(1-\xi)} - \frac{\Gamma(2-\beta)(\xi+(1-\xi)t)(1-s)(1-s)^{p-\beta-1}}{\Gamma(p-\beta)(1-\xi)},\ 0 \le s \le t < 1\ . \end{cases} \quad (12)$$

Putting the values of parameter ξ, β, α in (11), we get

$$G(t,s) = \begin{cases} \frac{.5(t-s)^{.5}+.5(1-s)^{.5}}{\Gamma 1.5(.5)} - \frac{\Gamma.5(.5+.5t)(1-s)^{0}}{\Gamma(1).5},\ s \le t, \\ \frac{.5(t-s)^{.5}+.5(1-s)^{.5}}{\Gamma 1.5(.5)} - \frac{\Gamma.5(.5+.5t)(1-s)^{0}}{\Gamma(1).5}, t \le s. \end{cases} \qquad (13)$$

From (13), one can easily obtain that $G^{*} \le \frac{\int_0^1 (1-s)^{.5}ds}{.5\Gamma(1.5)} + \frac{\Gamma(1.5)}{.5}$

which implies that $2kG^{*} \le 2\left(\frac{1}{11}\right)(4.16315) = 0.7569336 < 1$

and $\frac{2k\left(\Gamma(3-p)\Gamma(p-\beta+1)+\Gamma(2-\beta)\right)}{\Gamma(3-p)\Gamma(p-\beta+1)} = \frac{2}{11}\left(\frac{\Gamma(1.5)\Gamma(2)+\Gamma(1.5)}{\Gamma(1.5)\Gamma(2)}\right) = \frac{4}{11} < 1.$

Clearly it is obvious that

$\max\left\{2kG^{*}, \frac{2k\left(\Gamma(3-p)\Gamma(p-\beta+1)+\Gamma(2-\beta)\right)}{\Gamma(3-p)\Gamma(p-\beta+1)}\right\} < 1.$ Hence by Theorem 3.1, the given boundary value problem (11) has a unique solution on $J \in [0,1]$.

5. CONCLUSION

With the help of nonlinear Leray-Schauder fixed-point theorem, necessary and sufficient conditions were successfully developed which guarantee the existence of at least one solution. Further, the conditions for uniqueness were also obtained. Moreover if the parameter xi=-1 in the boundary and initial condition then the problem become anti periodic boundary value problem which is of more interest. For the readers interest here we pose some problems. The first case is to consider the same problem (1) while taking the p-Laplacian operator. For this case we may obtain some nice results. The outline of the future paper will like this; some other fixed point theorems need to be used for the existence of solution of main result. Moreover, we may check the stability, and other dynamical aspects the problems as well.

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