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A Note on Proper Homothetic Vector Fields of Reboucus Space-time by Using Lorentzian Manifolds

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Abstract: In this work direct integration and algebraic techniques have been implied to study the proper homothetic vector filds of Reboucus space-time. It has been proved that the above space-time does not admit any proper homothetic field. Its homothetic vector fields are the Killing vector fields. The Killing Lie algebras in this case have dimensions 3.

Keywords: Killing vector fields, proper homothetic vector fields, direct integration technique.

1. INTRODUCTION

The main objective of this work is to discuss the proper homothetic vector fields of Reboucus space-time by using direct integration technique. Here four dimensional space-time manifold with Lorentz metric g of signature (-, +, +, +) is denoted by M. The curvature tensor and Ricci tensor associated with g_{ab}, through the Levi-Civita connection, are denoted in component form by $R^{a}{}_{bcd}$ and

$R_{ab} = R^{c}{}_{acb}$. respectively.

The symbol L, semicolon and comma denote the Lie ,covariant and partial derivatives respectively. Symmetrization and skew-symmetrization are denoted by round and square brackets , respectively. On the manifold M, the covariant derivative of any vector field X can be splited as

$$X_{a;b} = \frac{1}{2}h_{ab} + G_{ab}, \qquad (1.1)$$

where $G_{ab}(=-G_{ba})$ is a skew symmetric and $h_{ab}(=h_{ba}) = L_X g_{ab}$ is a symmetric tensor on the manifold M, respectively. If $h_{ab} = 2\alpha g_{ab}, \alpha \in R$ Equivalently,

$$g_{ab,c}X^{c} + g_{bc}X^{c}{}_{,a} + g_{ac}X^{c}{}_{,b} = 2\alpha g_{ab} \qquad (1.2)$$

then X is said to be *homothetic* (and becomes *Killing* if $\alpha = 0$).

The vector field X is said to be proper homothetic. if it is not Killing vector field (Shabbir and Ramzan 2007). Further consequences and geometrical interpretation of (1.2) are explored in (Hall 2004). It also follows from (Hall and Steel 1990, Hall 1990)

$$L_X R^{a}{}_{bcd} = 0 \qquad L_X R_{ab} = 0 \qquad (1.3)$$

In terms of Lie brackets, the Lie algebra of a set of vector fields on a manifold is completely characterized by the structure constant $C^{a}{}_{bc}$ given by(Shabbir and Ramzan 2007) (Wald, 1984).

$$\left[X_b, X_c\right] = C^{a}_{bc}X_a, \quad C^{a}_{bc} = -C^{a}_{cb}, \qquad (1.4)$$

Where X_a shows the generator.

2. RESULTS

Consider a static Reboucus symmetric static space-time in the usual cylindrical coordinate system (t, r, θ, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with line element (Krori, et *al.*,1988)

$$ds^2 = -dt^2 + dr^2 + f(r)d\theta^2 + dz^2 + 2g(r)dtd\theta, \qquad (2.1)$$

Where $f(r) = -(1+3\cosh^2 2r)$ and $g(r) = \cosh 2r$. The above space-time admits three linearly independent Killing vector fields which are (Krori, et *al.*,1988) Shabbir and Amur 2006)

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}. \qquad (2.2)$$

A vector field X is said to be homothetic vector field if it satisfies the equation (1.2). One can write (1.2) explicitly using (2.1) as (Shabbir and Ramzan 2007)

$$X^{0}_{,0} - g(r)X^{2}_{,0} = \alpha, \qquad (2.3)$$

$$X^{1}_{,0} - X^{0}_{,1} + g(r)X^{2}_{,1} = 0, \qquad (2.4)$$

$$\frac{g'(r)}{2g(r)}X^{1} + \frac{1}{2}X^{0}_{,0} - \frac{1}{2g(r)}X^{0}_{,2} + \frac{f(r)}{2g(r)}X^{2}_{,0} + \frac{1}{2}X^{2}_{,2} = \alpha, \qquad (2.5)$$

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$$X^3_{,0} - X^0_{,3} + g(r) X^2_{,3} = 0, \tag{2.6}$$

$$X^1_{,1} = \alpha, \tag{2.7}$$

$$g(r) X^0_{,1} + f(r) X^2_{,1} + X^1_{,2} = 0, \tag{2.8}$$

$$X^3_{,1} + X^1_{,3} = 0, \tag{2.9}$$

$$\frac{f'(r)}{2f(r)} X^1 + \frac{g(r)}{f(r)} X^0_{,2} + X^2_{,2} = \alpha, \tag{2.10}$$

$$X^3_{,2} + g(r) X^0_{,3} + f(r) X^2_{,3} + = 0, \tag{2.11}$$

$$X^3_{,3} = \alpha, \tag{2.12}$$

where prime shows the derivative with respect to r . Equations (2.7) and (2.12) give

$$X^1 = \alpha r + F^1(t,\theta,z), \qquad X^3 = \alpha z + F^2(t,r,\theta).$$

Now using equations (2.3) and (2.4), one can find

$$X^0 = \frac{g(r)}{g'(r)}\left[F^1_t(t,\theta,z) - F_r^{\,3}(r,\theta,z),\right]$$
$$+\alpha t + F^3(r,\theta,z),$$
$$X^2 = \frac{1}{g'(r)}\left[F^1_t(t,\theta,z) - F_r^{\,3}(r,\theta,z),\right],$$

Thus we have the following system

$$\left.\begin{aligned} X^0 &= \frac{g(r)}{g'(r)}\left[F^1_t(t,\theta,z) - F_r^{\,3}(r,\theta,z),\right] \\ &+\alpha t + F^3(r,\theta,z), \\ X^1 &= \alpha r + F^1(t,\theta,z), \\ X^2 &= \frac{1}{g'(r)}\left[F^1_t(t,\theta,z) - F_r^{\,3}(r,\theta,z),\right], \\ X^3 &= \alpha z + F^2(t,r,\theta). \end{aligned}\right\} \tag{2.13}$$

Where $F^1(t,\theta,z), F^2(t,r,\theta), F^3(r,\theta,z)$ are the functions of integration and can be determined by solving the remaining equations. Using equation (2.13) in (2.9) and simplification gives

$F^1(t,\theta,z) = zG^1(t,\theta) + G^2(t,\theta)$ and

$F^2(t,r,\theta) = -rG^1(t,\theta) + G^3(t,\theta)$,where

$G^1(t,\theta), G^2(t,\theta)$ and $G^3(t,\theta)$ are the functions of integration. Substituting the above information in (2.13), we get the following system

$$\left.\begin{aligned} X^0 &= \frac{g(r)}{g'(r)}\left[zG^1_t(t,\theta) + G^2_t(t,\theta) - F^3_r(r,\theta,z),\right] \\ &+\alpha t + F^3(r,\theta,z), \\ X^1 &= \alpha r + zG^1(t,\theta) + G^2(t,\theta), \\ X^2 &= \frac{1}{g'(r)}\left[zG^1_t(t,\theta) + G^2_t(t,\theta) - F^3_r(r,\theta,z),\right], \\ X^3 &= \alpha z - rG^1(t,\theta) + G^3(t,\theta). \end{aligned}\right\} \tag{2.14}$$

Using equation (2.14) in (2.6) and then differentiating with respect to t and r respectively, after doing some calculation, we get $G^1(t,\theta) = t\,H^1(\theta) + H^2(\theta)$, $G^3(t,\theta) = t\,H^3(\theta) + H^4(\theta)$ and $F^3(r,\theta,z) = -rzH^1(\theta) + zH^3(\theta) + G^4(r,\theta)$. Using the above information in (2.14), we have the following system

$$\left.\begin{aligned} X^0 &= \frac{g(r)}{g'(r)}\left[2zH^1(\theta) + G_t^{\,2}(t,\theta) - G_r^{\,4}(r,\theta)\right] \\ &+\alpha t - rzH^1(\theta) + zH^3(\theta) + G^4(r,\theta), \\ X^1 &= \alpha r + tzH^1(\theta) + zH^2(\theta) + G^2(t,\theta), \\ X^2 &= \frac{1}{g'(r)}\left[2zH^1(\theta) + G_t^{\,2}(t,\theta) - G_r^{\,4}(r,\theta)\right], \\ X^3 &= \alpha z - rtH^1(\theta) - rH^2(\theta) + tH^3(\theta) \\ &+H^4(\theta). \end{aligned}\right\} \tag{2.15}$$

Now differentiating equations (2.8) and (2.11) with respect to z and r respectively, then subtracting we have

$$X^3_{,12} + g'(r) X^0_{,3} + f'(r) X^2_{,3} - X^1_{,23} = 0 \tag{2.16}$$

Substituting (2.15) in (2.16) and again differentiating resultant equation with respect to t ,we get

$H^1{}_\theta(\theta) = 0 \Rightarrow H^1(\theta) = c_1$. Backward substitution will give us:

$$2H^2(\theta) - g'(r)H^3(\theta) + \left\{rg'(r) - 2g(r) - 2\frac{f'(r)}{g'(r)}\right\}c_1 = 0 \tag{2.17}$$

Now differentiating equations (2.17) with respect to θ and r respectively, and after some calculation we have $H^3(\theta)=c_2$; $g''(r)\neq 0$ *and* $H^2(\theta)=c_3\theta+c_4$, where $c_2,c_3,c_4\in R.$ Substituting the above information in (2.17) and differentiation with respect to r gives

$$\left\{\frac{1}{g'(r)}[rg'(r)-2g(r)-2\frac{f'(r)}{g'(r)}]\right\}' c_1=0 \quad (2.18)$$

From equation (2.18) we see that the coefficient of c_1 can not be zero, so we conclude that $c_1=0$. Using all the above information in equation (2.15), we have the following system.

$$\left.\begin{aligned} X^0 &= \frac{g(r)}{g'(r)}\left[G_t^{\,2}(t,\theta)-G_r^{\,4}(r,\theta)\right] \\ &+\alpha t+G^4(r,\theta), \\ X^1 &= \alpha r+zc_4+G^2(t,\theta), \\ X^2 &= \frac{1}{g'(r)}\left[G_t^{\,2}(t,\theta)-G_r^{\,4}(r,\theta)\right], \\ X^3 &= \alpha z-rc_4+H^4(\theta). \end{aligned}\right\} \quad (2.19)$$

Using equation (2.19) in (2.8) and after some Calculation, we come to know that

$$\left\{(\frac{g(r)}{g'(r)})'g(r)+f(r)[\frac{1}{g'(r)}]'\right\}' G_{tt}^{\,2}(t,\theta)=0. \quad (2.20)$$

Again the coefficient of (2.20) cannot be zero, so we have $G_{tt}^{\,2}(t,\theta)=0\Rightarrow G^2(t,\theta)=tH^5(\theta)+H^6(\theta)$ .Now refreshing equation (2.19), we have

$$\left.\begin{aligned} X^0 &= \frac{g(r)}{g'(r)}\left[H^5(\theta)-G_r^{\,4}(r,\theta)\right] \\ &+\alpha t+G^4(r,\theta), \\ X^1 &= \alpha r+zc_4+tH^5(\theta)+H^6(\theta), \\ X^2 &= \frac{1}{g'(r)}\left[H^5(\theta)-G_r^{\,4}(r,\theta)\right], \\ X^3 &= \alpha z-rc_4+c_5 \end{aligned}\right\}. \quad (2.21)$$

Differentiation of equation (2.5) with respect to t, we get

$$\frac{g(r)}{g'(r)}H^5(\theta)=0\Rightarrow H^5(\theta)=0;\frac{g(r)}{g'(r)}\neq 0.$$

Now equation (2.21) takes the form

$$\left.\begin{aligned} X^0 &= -\frac{g(r)}{g'(r)}G_r^{\,4}(r,\theta)+\alpha t+G^4(r,\theta), \\ X^1 &= \alpha r+zc_4+H^6(\theta), \\ X^2 &= -\frac{1}{g'(r)}G_r^{\,4}(r,\theta), \\ X^3 &= \alpha z-rc_4+c_5 \end{aligned}\right\} \quad (2.22).$$

Substituting equation (2.22) in (2.8) and performing some lengthy calculations we attain the situation that $G_r^{\,4}(r,\theta)=0\Rightarrow G^4(r,\theta)=H^7(\theta)$ and $H_\theta^{\,6}(\theta)=0\Rightarrow H^6(\theta)=c_7$, these information reduce the equation (2.22) in the form

$$\left.\begin{aligned} X^0 &= \alpha t+H^7(\theta), \\ X^1 &= \alpha r+zc_4+c_6, \\ X^2 &= 0, \\ X^3 &= \alpha z-rc_4+c_5, \end{aligned}\right\} \quad (2.23)$$

Taking derivative of equation (2.10) with respect to θ,we have $H_{\theta\theta}^{\,7}(\theta)=0\Rightarrow\theta c_7+c_8$. Now the system (2.23) can be written as

$$\left.\begin{aligned} X^0 &= \alpha t+\theta c_7+c_8, \\ X^1 &= \alpha r+zc_4+c_6, \\ X^2 &= 0, \\ X^3 &= \alpha z-rc_4+c_5, \end{aligned}\right\} \quad (2.24)$$

Use of equation (2.24) in equation (2.10) and after doing some calculation, we have

$$\frac{2g'(r)g(r)+f(r)}{2f(r)g'(r)-f'(r)g(r)}c_7=\alpha \quad (2.25)$$

Differentiating (2.25) with respect to r, we get

$$\left[\frac{2g'(r)g(r)+f(r)}{2f(r)g'(r)-f'(r)g(r)}\right]' c_7=0. \quad (2.26)$$

Clearly the coefficient of c_7 can not be zero because $f(r)=-(1+3\cosh^2 2r)$ and $g(r)=\cosh 2r$, we have the only possibility that $c_7=0$. Substituting the value of c_7 in equation we get $\alpha=0$, which shows that there exist no proper homothetic vector fields. Hence we conclude that the homothetic vector fields in the above spacetime are the Killing vector fields which are given below

$$\left.\begin{aligned} X^0 &= c_8, \\ X^1 &= 0, \\ X^2 &= 0, \\ X^3 &= c_5, \end{aligned}\right\} \qquad (2.27)$$

The generators in this case are

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial \theta} \, and \, X_3 = \frac{\partial}{\partial z}.$$

3. CONCLUSION

In this paper a study of proper homothetic vector a field of Reboucus space-time is given. An approach is adopted to study the above space-time is direct integration technique (Shabbir and Ramzan 2007, Shabbir and Amur 2006), which suggested that no proper homothetic vector fields exist. From the above study it has been shown that homothetic vector fields in the above space time are the Killing vector fields.

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