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A Meshless collocation method for the approximate solution of a partial integro-differential equation

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Abstract: In this paper, a radial basis functions based collocation technique is developed to approximate solution of a partial integro-differential equation having a weakly singular kernel. The time derivative is approximated using backward finite difference approach while radial basis functions are collocated of over the spatial domain. Two types of radial basis functions namely Multiquadric and quantic are employed. Accuracy of the proposed method is evaluated in terms of L_2 and L_∞ , and by taking various time step sizes and different number of collocation points in the spatial domain. Numerical results of four test problems from literature are provided to study the accuracy and applicability of this method. Sufficient simulations are done to validate the method. Comparison of the results is also made with an existing method from literature.

Keywords: Collocation method, Radial basis function, Partial integro-differential equation, Weakly singular kernel.

1. INTRODUCTION

Mathematical modeling of many physical phenomenaleadsto functional equations. Such functional equations can be ordinary differential equation (ODE), partial differential equations (PDE), integral and integral-differential equation (IDE), partial integro-differential equation (PIDE) and stochastic equations etc. In fact ODEs/PD Esareunable to handle the hereditary property and memory of underlying model. Thus an integral term is introduced in the mathematical model of ODE or PDE to overcome this problem. As a result an equation is obtained which contains an integral term and is calledintegro-differential equation or a partial integro-differential equation. Various partial integro-differential equations have been obtained to study different physical phenomena including heat conduction (Miller, 1978), viscoelasticity (Lodge *et al.,* 1985) and convection-diffusion (Siddiqi and Arshad, 2013, Ali *et al.,*2015).

Several techniques have been developed for numerical solution of PIDEs. These include difference scheme (Macros, 1990), finite difference methods (Tang, 1993, Sheng and Tang, 1995, Luo, 2005, Jangveladze *et al.,* 2009), Galerkin approximation (Amiya *et al.,*2015), quasi-wavelet based numerical methods (Long *et al.,* 2012), finite element method (Chen *et al.*, 1999), B-splinecollocation methods (Zhang *et al.,* 2013, Siddiqi and Arshad, 2013, Ali *et al.,* 2015, Ahmad *et al.,* 2015 and Ali *et al.,* 2016). In this paper we construct a collocation method based on radial basis function(RBF) for convection-diffusion PIDE with a weakly singular kernel (Siddiqi and Arshad,2013, Ali *et al.,* 2015).RBF based methods have gained popularity in numerical techniques due to their mesh free nature, high rate of convergence, spatial independence and ease of implementation in complex geometries.In 1990, Kansa initiated the radial basis function collocation method to obtain approximate solution of PDE (Kansa, 1990). Later on, the method was extended by various author for the solution of different types of PDEs including (Sarler and Vertnik, 2006, Dehghan and Shokri, 2007, Siraj-ul-Islam *et al.,* 2010, Ali *et al.,* 2014). Recently, this method has been used for the solution of parabolic type PIDEs (Avazzadeh*et al.,*2012, Biazar and Asad, 2015).

2. CONSTRUCTIONOFTHE PROPOSED METHOD

Consider the following convection-difffusionintegro-differential equation(Siddiqi and Arshad, 2013, Ali *et al.,*2015):

$$w_t + mw_\xi - pw_{\xi\xi} = \int_0^t \mathrm{K}(t-q)w(\xi,q)dq + f(\xi,t),\ \beta_1 \le \xi \le \beta_2,\ t > 0, \quad (1)$$

subject to the initial condition:

$$w(\xi,0) = g(\xi),\ \beta_1 \le \xi \le \beta_2, \quad (2)$$

and boundary conditions:

$$w(\beta_1,t) = h_1(t),\quad w(\beta_2,t) = h_2(t),\ t > 0, \quad (3)$$

where$w = w(\xi,t)$ and$m > 0, p > 0$ are constants representing quantities of the convection and diffusion process. Here the weakly singular kernel is given by

$$\mathrm{K}(t-q) = (t-q)^{-\alpha}, \quad 0 < \alpha < 1.$$

Let$t_n = nk$, where k is the time step and $t_{n+1} = t_n + k, n = 0,1,2\ldots$. And$w^{n+1}(\xi) =$

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$w(\xi, t_{n+1}), f^{n+1}(\xi) = f(\xi, t_{n+1})$.The time derivative in Eq. (1) is discretized by backward difference formula at time level$n+1$as follows:

$w_t^{n+1}(\xi) \approx \frac{w^{n+1}(\xi) - w^n(\xi)}{k}$.(4)

Substituting Eq. (4) in Eq. (1), we get

$\frac{w^{n+1}(\xi) - w^n(\xi)}{k} + mw_\xi^{n+1}(\xi) - pw_{\xi\xi}^{n+1}(\xi) = \int_0^{t_{n+1}} \mathrm{K}(t_{n+1} - q)w(\xi, q)dq + f^{n+1}(\xi)$.(5)

The integral term in Eq. (5) is approximated as givenin (Ali *et al.*,2015):

$$\int_0^{t_{n+1}} \mathrm{K}(t^{n+1} - q)w(\xi, q)dq \approx \sum_{j=0}^{n} w^{n-j+1}(\xi)[(j+1)^{1-\alpha} - j^{1-\alpha}].(6)$$

From Eq. (6) in Eq. (5), we get

$\frac{w^{n+1}(\xi) - w^n(\xi)}{k} + mw_\xi^{n+1}(\xi) - pw_{\xi\xi}^{n+1}(\xi) = \frac{k^{1-\alpha}}{1-\alpha}\sum_{j=0}^{n} b_j w^{n-j+1}(\xi) + f^{n+1}(\xi)$.(7)

Where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 1, 2, 3, \ldots$

Eq. (7) can be rearranged as

$w^{n+1}(\xi) + mkw_\xi{}^{n+1}(\xi) - pkw_{\xi\xi}{}^{n+1}(\xi) - \frac{k^{2-\alpha}}{1-\alpha}w^{n+1}(\xi) = w^n(\xi) + \frac{k^{2-\alpha}}{1-\alpha}\sum_{j=1}^{n} b_j w^{n-j+1}(\xi) + kf^{n+1}(\xi), \quad n \geq 1$.(8)

Next we proceed as follows:

To construct the radial basis function collocation method, we choose $N+1$ distinct points ξ_i, $i = 0,1,2, \ldots, N$, from the interval $[a, b]$ such that $a = \xi_0$ and $\xi_N = b$. The RBF approximation of the function$w(\xi, t)$ at nth time level is given by

$$w^n(\xi) = \sum_{j=0}^{N} \lambda_j^n \psi(|\xi - \xi_j|), \quad (9)$$

where$\lambda_j^n$are unknown time dependent parameters to be determined from collocation conditions,$\psi$is a radial basis function and $|\xi - \xi_j|$ is Euclidean distance.From Eqs. (8)and (9),and collocating Eq. (8) at$\xi = \xi_i, i = 1,2, \ldots, N-1$, we get

$\sum_{j=0}^{N} \lambda_j^{n+1}\psi(|\xi_i - \xi_j|) + mk\sum_{j=0}^{N} \lambda_j^{n+1}\psi'(|\xi_i - \xi_j|) - pk\sum_{j=0}^{N} \lambda_j^{n+1}\psi''(|\xi_i - \xi_j|) - \frac{k^{2-\alpha}}{1-\alpha}\sum_{j=0}^{N} \lambda_j^{n+1}\psi(|\xi_i - \xi_j|) = \frac{k^{2-\alpha}}{1-\alpha}\left[\sum_{l=1}^{n} b_l \sum_{j=0}^{N} \lambda_j^{n-l+1}\psi(|\xi_i - \xi_j|)\right] + \sum_{j=0}^{N} \lambda_j^n \psi(|\xi_i - \xi_j|) + kf^{n+1}(\xi_i)$. (10)

Alsofrom Eqs.(3) and (9), we have

$$\sum_{j=0}^{N} \lambda_j^n \psi(|\xi_0 - \xi_j|) = h_1(t_{n+1}),$$

And$\sum_{j=0}^{N} \lambda_j^n \psi(|\xi_N - \xi_j|) = h_2(t_{n+1})$.(11)

In matrix form Eqs. (10) and (11)can be expressed as

$\boldsymbol{\lambda}^{n+1} = \mathbf{D}^{-1}\left[\frac{k^{2-\alpha}}{1-\alpha}\mathbf{A}\sum_{j=1}^{n} b_j \boldsymbol{\lambda}^{n-j+1} + \mathbf{A}\boldsymbol{\lambda}^n + k\boldsymbol{F}^{n+1}\right]$,(12)

where$\mathbf{D} = \left[\left(1 - \frac{k^{2-\alpha}}{1-\alpha}b_0\right)\mathbf{A} + mk\mathbf{B} - pk\mathbf{C}\right], \mathbf{A} = [a_{ij}]_{i,j=0}^{N}, \mathbf{B} = [b_{ij}]_{i,j=0}^{N}, \mathbf{C} = [c_{ij}]_{i,j=0}^{N}$such that$a_{ij} = \psi(|\xi_i - \xi_j|), b_{ij} = \begin{cases} \psi'(|\xi_i - \xi_j|), 1 \leq i \leq N-1, 0 \leq j \leq N, \\ 0, \quad i = 0, N, \ 0 \leq j \leq N. \end{cases}$

$c_{ij} = \begin{cases} \psi''(|\xi_i - \xi_j|), 1 \leq i \leq N-1, 0 \leq j \leq N, \\ 0, \quad i = 0, N, \ 0 \leq j \leq N. \end{cases}and\boldsymbol{\lambda}^n = [\lambda_0^n, \lambda_1^n, \ldots, \lambda_N^n]^T$. Inevitability of the matrix **D** is yet to be proved. Eq. (12) represents a system of $N+1$ equations in$N+1$ unknown parameters λ_j^{n+1}'s. Solving this system at each time level for the vector $\boldsymbol{\lambda}^{n+1}$ and then the approximate solution is obtained by using Eq. (9).

3. NUMERICAL TESTS

This section is dedicatedto sometest problems in order to test the RBF collocation method (12) for the solution of the problem given in Eqs. (1)-(3). The following two types of radial basis functions are used for computation:

Multiquadric$\psi(|\xi - \xi_j|) = \sqrt{|\xi - \xi_j|^2 + c^2}$

Quintic$\psi(|\xi - \xi_j|) = |\xi - \xi_j|^5$

The multiquadrictype RBF contains the parameter c which is known as shape parameter. This parameter has key role in obtaining accurate solution as well as stabilityofinterpolation matrix. To find that value of c which leads to optimal accuracy is still an open problem. Further discussion regarding c can be seen in (Siraj-ul-Islam *et al.,* 2010).The quintic type RBF is free of shape parameter. All computations are done using multiquadric except where quantic is mentioned specifically. For the sake of comparison all test problems are taken from the reference (Siddiqi and Arshad, 2013).

Test Problem 1: Consider Eq. (1)-(3) with the parameters $m = 0.05, \ p = 0.4, \ \alpha = \frac{1}{2}$and choose $f(\xi, t)$ so that the exact solution is $w(\xi, t) = (t+1)^2 \sin \pi\xi, \ 0 \leq \xi \leq 1$.

The initial and boundary conditions are obtained from the exact solution. Computations are performed with $N = 50, 100, k = 10^{-4}, 10^{-5}$, and the error norms L_∞ and L_2 are reported in **Tables 1-2** along with the results given in (Siddiqi and Arshad, 2013). Better accuracy of the RBF collocation method than the method of(Siddiqi and Arshad, 2013) is observed from **Tables 1-2. (Fig.1)**shows exact and approximate solutions obtained via the proposed method at$t = 1$ whereas **(Fig.2)**shows the RBF solutions at different time levelsin interval [0, 1].

Table 1:L_∞ and L_2 for Test Problem 1 when k=10^{-4}

			RBF		Cubic B-spline (Siddiqiand Arshad, 2013)	
N	c	M	L_∞	L_2	L_∞	L_2
100	0.11	10	5.08e-07	1.12e-08	9.43 e-06	1.32 e-07
		50	6.09e-07	3.97e-08	1.14e-05	3.31 e-07
		100	1.09e-06	7.77e-08	1.19 e-05	5.73 e-07
		500	5.88e-06	4.08e-07	3.53 e-05	2.43e-06
50	0.18	10	2.27e-06	6.73e-08	2.64e-05	5.62 e-07
		50	3.69e-06	1.43e-07	3.98e-05	1.08e-06
		100	4.04e-06	1.92e-07	4.40e-05	1.39e-06
		500	5.82e-06	5.53e-07	5.36e-05	2.86e-06
		10000	9.89e-05	9.44e-06	-----	----

Table 2:L_∞ and L_2 for Test Problem 1 when k=10^{-5}

			RBF		Cubic B-spline(Siddiqi and Arshad, 2013)	
N	c	M	L_∞	L_2	L_∞	L_2
100	0.09	10	3.80e-07	5.32e-09	4.38e-06	4.41e-08
		50	9.55e-07	1.49e-08	8.26e-06	9.59e-08
		100	1.12e-06	1.95e-08	9.56e-06	1.25e-07
		500	1.36e-06	3.26e-08	1.16e-05	2.25e-07
50	0.18	10	3.38e-07	9.60e-09	6.79e-06	1.38e-07
		50	1.50e-06	4.25e-08	2.00e-05	4.11e-07
		100	2.33e-06	6.84e-08	2.70e-05	5.87e-07
		500	3.71e-06	1.36e-07	4.02e-05	1.26e-06

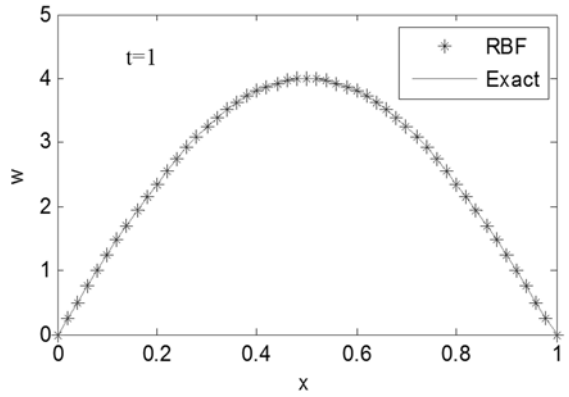


Fig. 1: Plot ofRBF and exact solutions for$N = 50, k = 10^{-4}$ for Test Problem 1

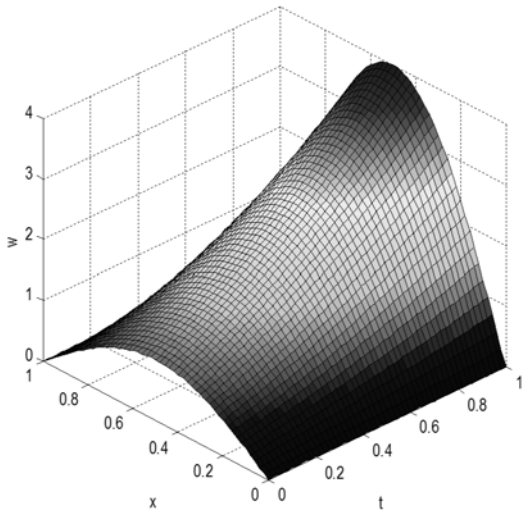


Fig.2: RBF solution over time interval [0, 1]using$N = 50$, $k = 10^{-4}$for Test Problem 1

Test Problem2: Consider Eq. (1) with the parameters $m = 0.1, p = 0.1, \alpha = \frac{1}{4}$ and $f(\xi, t)$is obtained so the exact solution is$w(\xi, t) = 2(t+1)\sin^2\xi,\ 0 \le \xi \le 4\pi$. The initial and boundary conditions are given as:
$w(\xi, 0) = 0, 0 \le \xi \le 4\pi,$
$w(0, t) = 0, w(1, t) = 0.$
Numerical results are obtained using $N = 10, 50,$ $k = 10^{-4}, 10^{-5}$.L_∞, L_2atdifferent values of $M$are given in Tables (3)-(4) along with the results of obtained by the method(Siddiqi and Arshad, 2013). Fig.3shows exact and RBF solution at $t = 1$. Fig.4 shows RBF solution over the time interval [0, 1].

Table 3: L_∞ and L_2 for Test Problem 2when k=10^{-4}

			RBF		Cubic B-spline (Siddiqi and Arshad, 2013)	
N	c	M	L_∞	L_2	L_∞	L_2
10	6.9	10	9.29e-05	1.06e-05	1.80e-04	3.65e-05
		50	4.65e-04	5.33e-05	8.172-04	1.66e-04
		100	9.33e-04	1.07e-04	1.50e-03	3.02e-04
50	2.1	10	3.63e-08	1.00e-09	6.40e-06	7.53e-07
		50	1.81e-07	5.04e-09	1.78e-04	1.49e-05
		100	3.61e-07	1.02e-08	6.83e-04	5.61e-05
		10000	8.38e-05	7.42e-06	----	----

Table 4: L_∞ and L_2 for Test Problem 2when k=10^{-5}

			RBF		Cubic B-spline (Siddiqi and Arshad, 2013)	
N	c	M	L_∞	L_2	L_∞	L_2
10	6.1	10	2.14e-06	3.60e-07	1.90e-05	3.78e-06
		50	1.07e-05	1.80e-06	9.33e-05	1.87e-05
		100	2.14e-05	3.60e-06	1.82e-04	3.68e-05
50	2.1	10	3.87e-09	1.08e-10	9.47e-07	7.30e-08
		50	1.95e-08	5.30e-10	4.23e-06	2.89e-07
		100	3.93e-08	1.06e-09	7.14e-06	6.35e-07

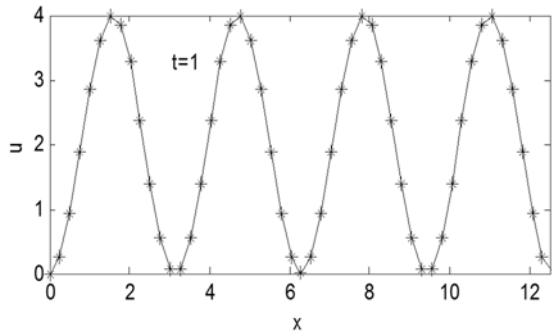


Fig.3: Plot ofRBF and exact solutions for$N = 50, k = 10^{-4}$ at $t = 1$for Test Problem 2

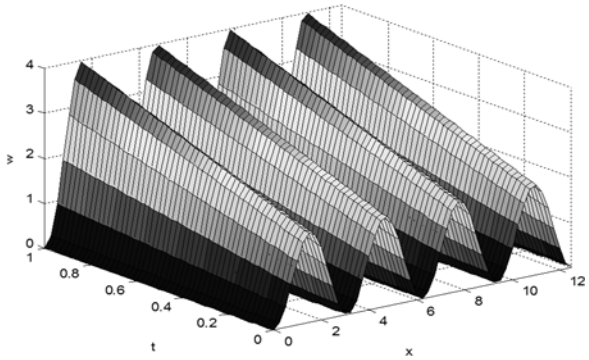


Fig.4: RBF solution over time interval [0, 1] using$N = 50$, $k = 10^{-4}$for Test Problem 2

Test Problem 3: Consider Eqs. (1)-(3) with the parameters $m = 0.5, p = 0.005, \alpha = \frac{1}{3}$ and choose $f(\xi, t)$in such a manner that the exact solution of the problem becomes$w(\xi, t) = (t + 1)\cos \pi\xi$.

The initial and boundary conditions are obtained from the exact solution.

Numerical simulations are carried outusing$N = 50, k = 10^{-4}, 10^{-5}$. L_∞, L_2atdifferent values of M are reported in Table 5 along with the results given(Siddiqi and Arshad, 2013). In Table 6, the error norms are found via quintic type of RBF. Plots of the RBF solutions are shown I n Figs. 5 and 6.

Table 5: L_∞ and L_2 for Test Problem 3when N=100.

			RBF		Cubic B-spline (Siddiqi and Arshad, 2013)	
k	c	M	L_∞	L_2	L_∞	L_2
10^{-4}	0.09	10	6.88e-08	7.64e-10	1.17e-06	3.01e-05
		50	4.00e-07	4.21e-09	2.01e-05	3.30e-04
		100	7.92e-07	8.36e-09	7.26e-05	1.06e-03
		500	1.59e-06	2.86e-08	1.50e-03	2.17e-02
		10000	3.70e-05	2.31e-06	----	----
10^{-5}	0.09	10	6.36e-09	7.29e-11	1.47e-08	3.76e-07
		50	3.26e-08	3.69e-10	2.54e-07	5.01e-06
		100	6.65e-08	7.44e-10	9.21e-07	1.68e-05
		500	4.19e-07	4.64e-09	1.91e-05	2.84e-04

Table 6: L_∞ and L_2 obtained through quantic RBF forTest Problem 3

		k=10^{-4}		k=10^{-5}	
N	M	L_∞	L_2	L_∞	L_2
50	10	1.14e-06	2.44e-08	1.14e-07	2.47e-09
	50	5.63e-06	1.17e-07	5.72e-07	1.23e-08
	100	1.10e-05	2.22e-07	1.14e-06	2.44e-08
	500	4.06e-05	8.75e-07	5.63e-06	1.17e-07

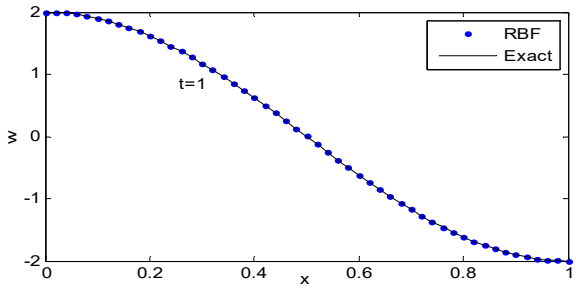


Fig.5: Plot ofRBF and exact solutions for$N = 50, k = 10^{-4}$ at $t = 1$for Test Problem 3

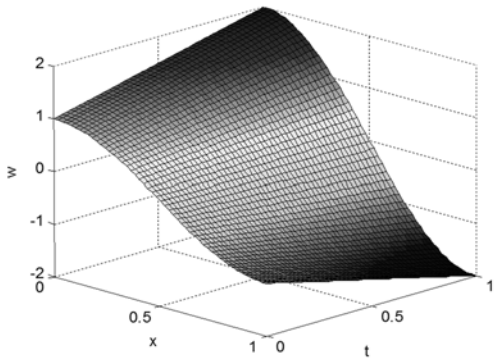


Fig.6: RBF solution over time interval [0, 1] using$N = 50$, $k = 10^{-4}$for Test Problem 3

Test Problem4: Now we consider Eq. (1) with the parameters $m = 0.005, p = 0.5, \alpha = \frac{1}{3}$and $f(\xi, t)$ is obtained by taking the exact solution of Eq. (1)-(3) as $w(\xi, t) = (t+1)^2\big((1-\cos 2\pi\xi) + 2\pi^2\xi(1-\xi)\big)$. (13) The initial and boundary conditions are extracted from Eq. (13). Simulations are performed with the above parameters using $N = 100,\ 50, k = 10^{-4}, 10^{-5}$and the results are noted in Tables 7-8.Plots of the RBF solutions are shown in Figs. 7 and 8.

Table 7: L_∞ and L_2 for Test Problem 4when k=10^{-4}

			RBF		Cubic B-spline (Siddiqi and Arshad, 2013)	
N	c	M	L_∞	L_2	L_∞	L_2
100	0.11	10	3.15e-06	7.40e-08	6.63e-06	5.08e-07
		50	3.70e-06	2.54e-07	3.14e-05	2.42e-06
		100	6.97e-06	4.84e-07	5.92e-05	4.64e-06
		500	3.40e-05	2.32e-06	2.29e-04	1.93e-05
50	0.25	10	1.79e-06	1.57e-07	2.43e-05	2.01e-06
		50	3.50e-06	3.48e-07	1.01e-04	8.88e-06
		100	7.01e-06	6.81e-07	1.72e-04	1.58e-05
		500	3.41e-05	3.30e-06	4.70e-04	4.18e-05
		10000	4.53e-04	4.49e-05	----	----

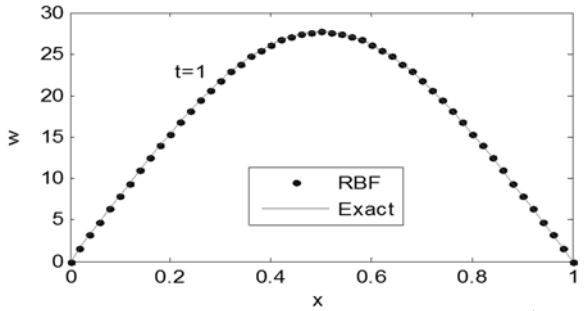


Fig.7: Plot ofRBF and exact solutions for$N = 50, k = 10^{-4}$ at $t = 1$for Test Problem 4

Table 8: L_∞ and L_2 for Test Problem 4when k=10^{-5}

			RBF		Cubic B-spline (Siddiqi and Arshad, 2013)	
N	c	M	L_∞	L_2	L_∞	L_2
100	0.1	10	2.93e-06	4.01e-08	1.90e-06	4.54e-08
		50	6.90e-06	1.04e-07	3.09e-06	2.01e-07
		100	8.02e-06	1.35e-07	5.96e-06	3.91e-07
		500	9.44e-06	2.22e-07	2.6e-05	1.75e-06
50	0.2	10	1.44e-06	4.04e-08	2.93e-06	2.48e-07
		50	6.32e-06	1.78e-07	1.25e-05	1.20e-06
		100	9.37e-06	2.77e-07	2.48e-05	2.35e-06
		500	1.39e-05	5.28e-07	1.19e-04	1.06e-05

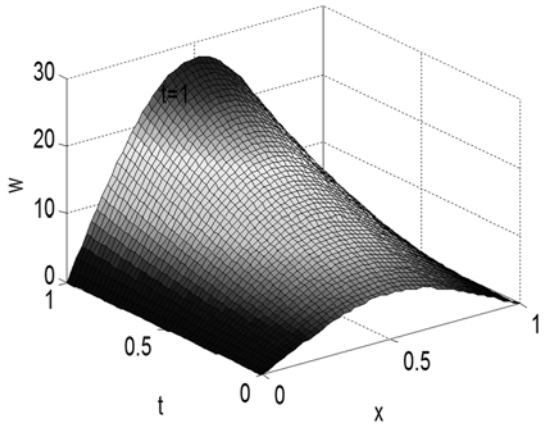


Fig.8: RBF solution over time interval [0, 1] using$N = 50$, $k = 10^{-4}$for Test Problem 4

4. CONCLUSION

In this paper, a radial basis functions based collocation principle for the so lution of convection-diffusionintegro-differential equation having a weakly singular kernel. From simulation, better accuracy is observed and the numerical results are also in good agreement with exact solution. The results are also compared with cubic B-spline collocation method. However, the numerical scheme requires small time step for higher accuracy.

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