Sindh Univ. Res. Jour. (Sci. Ser.) Vol. 47 (3) 545-548 (2015)

SINDH UNIVERSITY RESEARCH JOURNAL(SCIENCE SERIES)

Minimum Error Bounds for Local Truncation Errors to Iteratively Solve Scalar and Vector-Valued ODEs

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Received 17th May 2015 and Revised 08th August 2015

Abstract: This paper aims at finding minimum error bounds on step size for a few second order explicit iterative techniques. Three standard most widely used Runge-Kutta type techniques and two recently developed iterative techniques are selected for analysis. Computation and examination of principal local truncation error and error bounds on step length reveal that minimum error bound is attained by Heun, Modified Improved Modified Euler (MIME) as derived in (Ma, 2010) and proposed iterative techniques and the same is confirmed by both scalar and vector-valued numerical problems given. Further, proposed iterative technique is found to have contained smallest local truncation error. Numerical calculations have been carried out using MATLAB version 8.1 (R2013a) in double precision arithmetic.

Keywords: error bound, iterative, truncation error, Heun, Scalar and Vector-valued IVPs.

1. INTRODUCTION

Ordinary Differential Equations (ODEs) are considered to be the most significant tool to solve number of physical and natural problems belonging to the fields of Science, Business, Economics, Pharmacy, Chemistry and Engineering as discussed in (Chandio and Memon, 2010), (Kreyszig, 2011), and (OCHOCHE, 2007); just to mention a few. Unfortunately, solution of every type of ODE cannot be expressed in terms of basic mathematical functions and therefore one has to either find solution to ODEs in the form of special mathematical functions or seek iterative (approximate) solution using difference equations (approximate discrete models to ODEs) based on numerous iterative techniques as discussed in (Chandio, *et al.*, 2010).Many scholars have either improved existing iterative techniques or devised new schemes to solve scalar and vector-valued ODEs subject to certain conditions such as (Chandio and Memon, 2010), (Ibijola and Ogunrinde, 2010), (Fatunla, 1976), (Ibijola, 1997), (Ramos, 2007); to mention a few. Generally, the problem

$$\frac{d\vec{y}}{dx} = \vec{f}\left(x, \vec{y}\right); \ \vec{y}\left(x_0\right) = \vec{y}_0 \tag{1}$$

is termed as an initial – value problem; where $\vec{y}, \vec{f}$ and $\vec{y}_0$ are vector quantities with *n* elements. Study of error bounds is indispensable whenever an iterative technique is either improved or a new one is proposed. Without sound knowledge of error bounds on step length of the technique concerned; it is precarious to employ the user to choose the most efficient iterative technique from the pool of numerous techniques developed by Ma, (2010).

2. PROBLEM DESCRIPTION AND METHODOLOGY

Main focus of this research paper is to analyze principal error function and error bounds imposed on step length (Δt) for truncation error involved in standard explicit linear single-step Runge – Kutta (RK) type techniques and techniques recently improved and proposed by a few scholars. These explicit techniques would serve well at the cost of accuracy, that is; they are used when computational effort and time are of utmost importance. For the sake of simplicity; analysis is presented for second order techniques to solve only; however, can be generalized to higher order systems.

$$y'(x) = f\left(x, y(x)\right); y\left(x_0\right) = y_0$$

From the iterative techniques selected; optimum techniques are decided on the basis of principal error function and bound on truncation error introduced by (Lotkin, 1951). As discussed in (Burden and Faires, 2010); k^{th}-order RK technique is developed as

$$y_{i+1} = y_i + \Delta t \sum_{r=1}^{k} w_r k_r \tag{2}$$

where $k_r = f\left(x_i + \alpha_r \Delta t, y_i + \Delta t \sum_{s=1}^{r-1} \beta_{rs} k_s\right)$.

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For k^{th}-order RK technique to be consistent; the following condition is also met

$$\alpha_r = \sum_{s=1}^{r-1} \beta_{rs} k_s \ ; \ r = 2,3,\cdots,k$$

The leading truncation error term $\left(\Psi(x,y)\Delta t^{k+1}\right)$ satisfies

$$\left|\Psi(x,y)\Delta t^{k+1}\right| a\Delta t^{k+1} P^k Q \qquad (3)$$

A technique attaining minimum value of the parameter 'a' in (3) is regarded as optimum technique. For second

$$\left|\Psi(x,y)\Delta x^3\right| < \left(\frac{1}{3} + \left|\frac{2}{3} - \alpha_2\right|\right)\Delta x^3 P^2 Q$$

order explicit linear sin

gle-step methods; Lotkin bound (Lotkin, 1951) for the principal local truncation error

(PLTE) can be expressed as

where

$\Psi(x,y) =$ Principal Error Function

$\alpha_2 =$ Parameter from 2nd slope of the method concerned

$$\left|f(x,y)\right| < Q, \left|\frac{\partial f(x,y)}{\partial y}\right| < P \text{ and } \left|\frac{\partial^{p+q} f(x,y)}{\partial x^p \partial y^q}\right| < \frac{P^{p+q}}{Q^{q-1}}$$

with $(p+q) \leq$ order of the method.

On the basis of above description; bounds for the principal local truncation error have been found. Algorithms of the iterative techniques being used in current work are given below:

Midpoint Euler technique (source: (Burden and Faires, 2010)):

$$y_{i+1} = y_i + \Delta t\, f\left(x_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{2} f(x_i, y_i)\right)$$

Bound on PLTE: $\left|\Psi(x,y)\Delta x^3\right| < \frac{1}{2}\Delta x^3 P^2 Q$

Modified Euler's technique (source: (Burden and Faires, 2010)):

$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(x_i, y_i) + f\left(x_i + \Delta t, y_i + \Delta t\, f(x_i, y_i)\right)\right]$$

Bound on PLTE: $\left|\Psi(x,y)\Delta x^3\right| < \frac{2}{3}\Delta x^3 P^2 Q$

Heun's technique (source: (Burden and Faires, 2010)):

$$y_{i+1} = y_i + \frac{\Delta t}{4}\left[\begin{array}{l} f(x_i, y_i) + \\ 3f\left(x_i + \frac{2}{3}\Delta t, y_i + \frac{2}{3}\Delta t\, f(x_i, y_i)\right)\end{array}\right]$$

Bound on PLTE: $\left|\Psi(x,y)\Delta x^3\right| < \frac{1}{3}\Delta x^3 P^2 Q$

MIME's technique developed by OCHOCHE, (2008):

$$y_{i+1} = y_i + \Delta t\, f\left(\begin{array}{l} x_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{2} \\ f\left(x_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{2} f(x_i, y_i)\right)\end{array}\right)$$

Bound on PLTE[1]: $\left|\Psi(x,y)\Delta x^3\right| < \frac{1}{3}\Delta x^3 P^2 Q$

Iterative technique proposed by Qureshi *et al.,* (2013 and 2014):

$$y_{i+1} = y_i + \Delta t\, f\left(t_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{4}\left(\begin{array}{l} f(t_i, y_i) + \\ f\left(t_{i+1}, y_i + \Delta t\, f(t_i, y_i)\right)\end{array}\right)\right)$$

Bound on PLTE: $\left|\Psi(x,y)\Delta x^3\right| < \frac{1}{3}\Delta x^3 P^2 Q$

3. ANALYSIS AND FINDINGS

In this part of the work, scalar and vector – valued Initial Value Problems (IVPs) are considered to find error bounds on step length of the technique concerned whereas error tolerance has been set to $\varepsilon = 1\mathrm{E}-04$. All necessary computations and graphical displays have been carried out using MATLAB version 8.1 (R2013a) in double precision arithmetic. Consider a first order IVP:

Problem 01. $\left.\begin{array}{l} y'(x) = y\cos(x) \\ y(0) = 1,\ x \in [0,1] \end{array}\right\}$ (Source: (Edwards and Penney, 2008))

[1] (Ma, 2010) miscalculated the error bound for MIME technique; a correct version of which is given in the present work.

Exact Periodic Solution for problem 1 is given by $y(x) = e^{\sin(x)}$.

It is easy to check that

$$\max_{x\in[0,1]} \left| f(x,y) \right| < 1.458528535930293e{+}00$$

and $\max_{x\in[0,1]} \left| \frac{\partial f(x,y)}{\partial y} \right| < 1$.

Table 1: Comparison Of Step Size Error Bounds For Iterative Techniques Applied On Problem 01.

Technique	Step size error bound
Midpoint Euler	5.156697887634794e-02
Modified Euler	4.685170961317590e-02
Heun	5.902945516520230e-02
MIME	5.902945516520230e-02
Proposed	5.902945516520230e-02

In the above **(Table-1)** it is observed that iterative techniques of Heun's, MIME's and Proposed all have the same error bound on the step length for local truncation error (LTE) which is also the minimum error bound in the family of error bounds of explicit 2nd order RK type techniques. However, it is easy to be noted from the table that error bound of Midpoint and Modified Euler technique differ from each other by approximately 4.715269263172041e-03. Furthermore, absolute relative percent local and global truncation errors have been computed in **(Table 2)** for problem 01 at $x = 1$ with $\Delta t = 0.01$ [xpounding proposed method to have smallest LTE and the same is depicted by figure 01 whereas nothing can be said about global truncation errors (see figure 2) at this stage for the present work is concerned with discussion of error bounds for LTE, only.

Table 2: Absolute Relative Percent Local And Global Truncation Errors For Problem 01.

Technique	Local	Global
Midpoint Euler	1.092631178541352e-05	1.599837544445278e-04
Modified Euler	7.304510684644814e-06	1.768791989246086e-03
Heun	4.849620677792931e-06	4.821543159083185e-04
MIME	3.633973493591920e-06	8.826069472685852e-04
Proposed	3.584339816498499e-06	8.745877904174400e-04

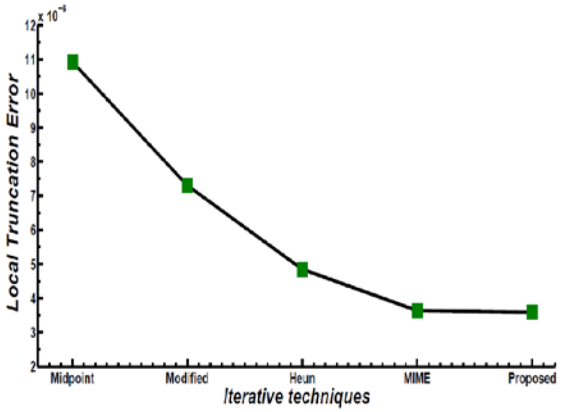


Fig. 1. Absolute Relative Percent Local Truncation Errors For Problem 01.

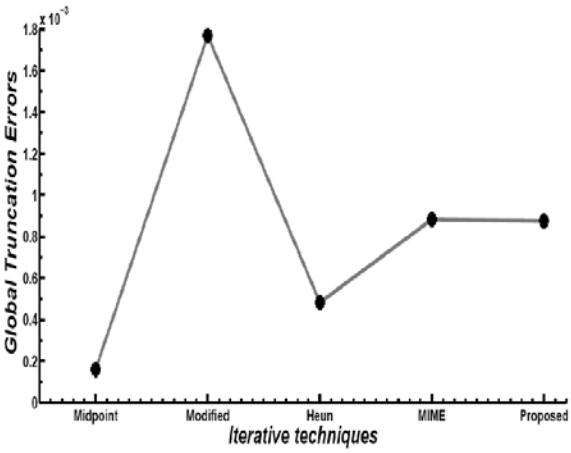


Fig. 2. Absolute Relative Percent Global Truncation Errors For Problem 01.

Next Problem Is A 1st Order Nonlinear Ivp That Is Given As"

$$\left.\begin{aligned} y'(x) = \sqrt{y}\ ;\ y(0) = 1 \\ x \in [0,1] \end{aligned}\right\}$$

Problem 02.

(Source: (Ma, 2010))

Exact Solution for problem 2 is given by $y(x) = 0.25(x+2)^2$,

where it is found that $P = 0.5$ and $Q = 1$. Performance of iterative techniques under consideration in connection with their step size error bounds is shown in figure 03. Based on numerical behavior of line graph in the figure; equivalence of Heun, MIME and proposed iterative techniques can easily be established.

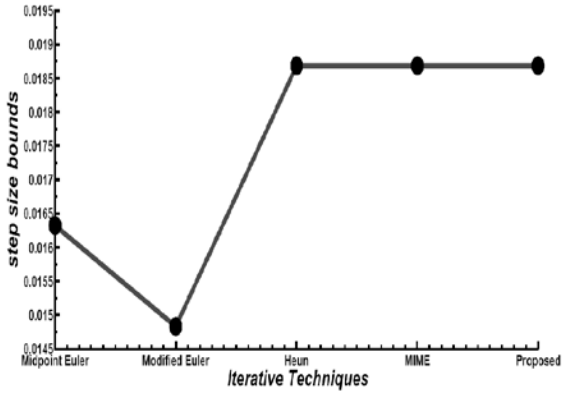


Fig. 3. Comparison Of Step Size Bounds For Iterative Techniques Applied On Problem 02.

4. DISCUSSION AND CONCLUSION

The focus of the paper was to analyze minimum error bound - under Lotkin's settings – imposed on step size of an explicit RK type iterative technique in order to choose a better technique to solve practical problems involving IVPs. On the basis of analysis and computations presented; equivalency in error bounds was found among Heun's, MIME and proposed techniques. Therefore, these three techniques are said to have minimum error bounds; that is, comparatively larger step size is allowed to be taken. A correct version of principal local truncation error for MIME technique was also presented.

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