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Analytical Solutions Under No Slip Effects for Accelerated Flows of Maxwell Fluids

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AbstractIn This research article, we present the analysis for velocity field and adequate shear stress of the Maxwell fluid moving with no slip effects over an infinite plate by using (integral transforms) Laplace and Fourier Sine transforms and their inverses. Analytical solutions are established for simple functions, integral and primitive forms and convolution product. All general solutions satisfy all necessary conditions (Initial and boundary conditions). Furthermore general solutions are reduced for the behavior of Newtonian fluid as the limiting case. In the last, all the solutions are presented graphically for comparison and thus concluded on their basis our facts and results.

Keywords*:* Maxwell fluid, No Slip Condition, Fourier Sine and Laplace Transforms, Exact solutions, Newtonian fluid, Graphical highlights.

1. INTRODUCTION

The fundamental equations in fluid Mechanics are the Navier Stokes equations. A few exact analytical solutions of Navier Stokes equations are not available because of the relevancy of nonlinearity. The importance of analytic solutions as compared for numerical solutions is huge because analytical solutions are valid and accurate on the whole domain of definition and the numerical solutions are only valid at particular points on the particular domain of definition. Analytical and exact solutions assist for approximations to several particular problems and to validate the numerical schemes and methods for complex flow problems. For checking the exactness, correctness and accuracy for numerical results, the exact solutions have got much attention for their abundant technological and industrial applications such as exotic lubricants, colloidal and suspension solutions, extrusion polymers fluid, food stuff and many other applications have been conducted by Zaman *et al.* (2013), Jamil *et al.* (2012) and Jamil *et al.* (2011). It is known fact that materials are characterized by their constitutive equations. The relationship (the simplest constitutive equation) between strain rate and stress is linear in Newtonian. Seeing the industrial importance and technological applications, Such as food stuffs, molten plastics, food industry, bio-engineering and oil have nonlinear relation. Above these items (materials) are followed by viscoelastic fluids. Lot of models has been displayed for viscoelastic behavior, due to the technological advancements in rate type and differential type fluids have gotten many attentions among the scientists and researchers which is studied in Jamil *et al.* (2013). In recent times, the powerful achievements in the viscoelasticity have been encountered in Jamil *et al.* (2012), Jamil *et al.* (2011) and Friedrich (1995). On the other hand, Maxwell fluid is a subclass of the rate type fluid that has relaxation phenomena. In the viscoelastic flows, for the large and small dimensionless relaxation time Maxwell fluid model has been applied hugely. Dealing with the Maxwell fluid, some recent investigations are given in references Fetecau *et al.* (2010), Corina *et al.* (2009), Najeeb *et al.* (2011), Najeeb *et al.* (2012) and Taswar *et al.* (2004).

Keeping in our acquaintance and knowledge, no study and search are existed yet in the literature to discuss the effects of no slip effects on the Maxwell fluid for incompressible flows of the rate type fluids. Seeing and viewing this fact the main aim of the present work is the analysis for no slip effects on the incompressible flows of a Maxwell fluid (a subclass of rate type fluids). The objective of current research paper is to derive analytical solutions of Maxwell fluid with effects of no slip condition. The general solutions are achieved by utilizing (mathematical transformations techniques) Fourier sine and Laplace transforms and their inverses, are expressed as the simple functions, integral forms and convolutions products. Associated solutions corresponding to the Newtonian fluids are found out as a limiting case as well as graphical descriptions are highlighted in detail to display the special limitations and effects of emergent flow parameters.

2. MATHEMATICAL MODEL

The constitutive equations for an incompressible Maxwell fluid in the Cauchy Stress tensor are in Corina *et al.* (2009) and Kashif *et al.* (2015)

$$\mathbf{T} = -\mathrm{p}\mathbf{I} + \mathbf{S}, \quad \boldsymbol{S} + \lambda \frac{\delta \boldsymbol{S}}{\delta t} = \mu \boldsymbol{A_1}, \qquad (1)$$

$$\frac{\delta \boldsymbol{S}}{\delta t} = \frac{d\boldsymbol{S}}{dt} - \boldsymbol{LS} - \boldsymbol{SL}^T = \boldsymbol{S} - \boldsymbol{LS} - \boldsymbol{SL}^T, \qquad (2)$$

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Where **T** is the Cauchy stress tensor, −p**I** represents the indeterminate spherical stress, **S** is the extra stress tensor, λ is the relaxation time, μ the dynamic viscosity, $\boldsymbol{A_1}$ is the first Rivlin Ericksen tensor and $\frac{\delta \boldsymbol{S}}{\delta t}$ the upper convected derivative. Under consideration of our problem, the constraint of incompressibility for flows is automatically satisfied

$$\boldsymbol{V} = u(y,t)\boldsymbol{i} = \boldsymbol{V}(y,t)\,, \boldsymbol{S} = \boldsymbol{S}(y,t), \quad (3)$$

For an incompressible Maxwell fluids performing the same motion in the absence of pressure gradient are governing equations

$$\nu \frac{\partial^2 u}{\partial^2 t} = \left(1 + \lambda \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}\,, \quad (4)$$

$$\mu \frac{\partial u}{\partial y} = \left(1 + \lambda \frac{\partial}{\partial t}\right)\tau. \quad (5)$$

Consider an incompressible Maxwell fluid occupying the space above a flat plate perpendicular to the *y*-axis. Initially, the fluid is at rest and at the moment $t = 0^+$ the plate is impulsively brought to the constant velocity *U* in its plane. Due to the tangential shear stress, the fluid above the plate is slowly and gradually moved. The relevant problem under initial and boundary conditions is **Y**

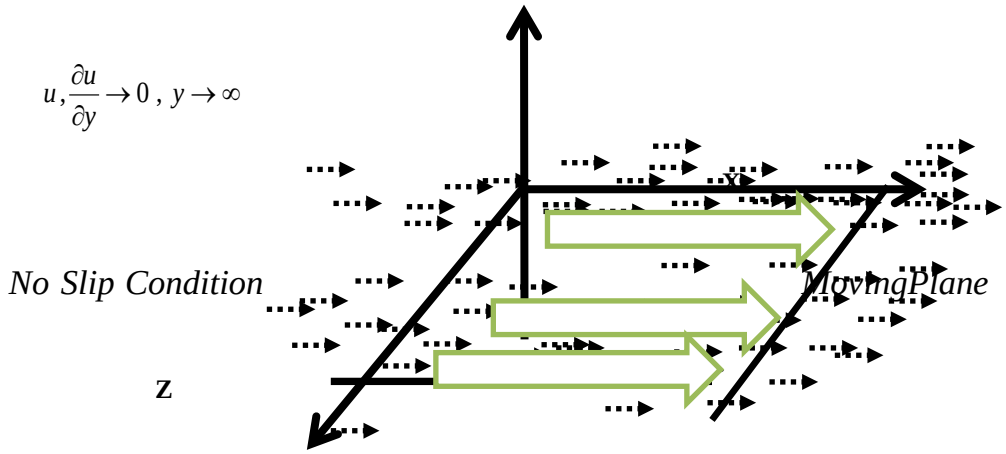


Fig. 1: Geometrical over view of the problem.

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0, \quad \tau(y,0) = 0, \quad y > 0, \quad (6)$$

$$u(0,t) = UH(t)t^m, \qquad m > 0, \qquad t \geq 0. \quad (7)$$

It is quite clear, naturally conditions are with $H(t)$ is the unit step function.

$$u(y,t), \quad \frac{\partial u(y,t)}{\partial t} \to 0 \; as \; y \to \infty \quad and \quad t > 0 \quad (8)$$

have to be also satisfied.

3. SOLUTION FOR THE PROBLEM

Calculating the velocity field, We use Fourier sine transforms Corina *et al.* (2002), Taswar*et al.* (2004), Najeeb *et al.* (2012), Zierep *et al.* (2007), Corina *et al.* (2010), Kashif *et al.* (2015) and Debnath *et al.* (2007), and keeping in concentration the initial and boundary conditions (6) and (7) , we get

$$\left\{-\nu\xi^2 u_s + \sqrt{\frac{2}{\pi}}\nu\xi\, UH(t)t^m\right\} = \left\{\frac{\partial u_s}{\partial t} + \lambda\frac{\partial^2 u_s}{\partial^2 t}\right\}, \quad (9)$$

Where $H(t)$ denotes the unit step function, and the initial conditions are satisfied for u_s Fourier sine transform

$$u_s = \frac{\partial u_s}{\partial t} = 0, \quad \xi > 0\,, \quad (10)$$

Having the Laplace transform to (9) and (10), we obtain

$$\overline{u_s} = \sqrt{\frac{2}{\pi}}\,\frac{m!\,U\,\xi\,\nu}{[\lambda q^2 + q + \nu\xi^2]q^{n+1}}\,, \quad (11)$$

decomposing equation (11) equivalently

$$\overline{u_s} = \frac{m!\;U}{\xi}\sqrt{\frac{2}{\pi}}\left[\frac{1}{q^{m+1}} - \frac{(1+\lambda q)}{q^m[\lambda q^2 + q + \nu\xi^2]}\right] \quad (12)$$

Using the following decomposition equation (12) can be expressed as

Where, the algebraic equation $\lambda q^2 + q + \nu\xi^2 = (q - q_1)(q - q_2)$ has to satisfy the following fact $q_1, q_2 = -\dfrac{(1) \pm \sqrt{(1)^2 - 4\lambda(\nu\xi^2)}}{2\lambda}$

$$\overline{u_s} = \frac{U\, m!}{\xi}\sqrt{\frac{2}{\pi}}\left[\frac{1}{q^{m+1}} - \frac{(1+\lambda q)}{\lambda q^m[(q-q_1)(q-q_2)\,]}\right], \quad (13)$$

Applying inverse Fourier sine formula on equation (13), we have

$$\bar{u} = \frac{2\ m!\ U}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\left[\frac{1}{q^{m+1}} - \frac{(1+\lambda q)}{\lambda q^m[(q-q_2)(q-q_1)]}\right]d\xi, \quad (14)$$

simplifying

$$\bar{u} = \frac{2\, m!\, U}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\left[\frac{1}{q^{m+1}} - \frac{1}{\lambda q^m}\left\{\frac{(1+\lambda q_1)}{(q_1-q_2)(q-q_1)} - \frac{(1+\lambda q_2)}{(q_1-q_2)(q-q_2)}\right\}\right]d\xi, \quad (15)$$

using the following fact of integral,

$$\int_0^\infty \frac{sin(y\xi)}{\xi}\, d\xi = \frac{\pi}{2}, \qquad y > 0$$

We have

$$\bar{u} = \frac{U\, m!}{q^{n+1}} - \frac{2\, U\, m!}{\pi\, \lambda}\int_0^\infty \frac{\sin(y\xi)}{\xi}\frac{1}{q^m}\left\{\frac{(1+\lambda q_1)}{(q_1-q_2)(q-q_1)} - \frac{(1+\lambda q_2)}{(q_1-q_2)(q-q_2)}\right\}d\xi, \quad (16)$$

Finally, we apply the inverse Laplace transform to (16), we find the velocity field in the convolution product

$$u = U\, H(t)t^m - \frac{2\, m\, U\, H(t)}{(q_1-q_2)\,\lambda\,\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\, t^{m-1} * \{e^{q_1 t}(1+q_1\lambda) - e^{q_2 t}(1+q_2\lambda)\}\, d\xi, \quad (17)$$

Employing the convolution theorem on (17), we find the simple expression in the integral form for velocity field

$$u = UH(t)t^m - \frac{2\, m\, UH(t)}{(q_1-q_2)\lambda\pi}\int_0^\infty \int_0^t \frac{\sin(y\xi)}{\xi}\, t^{m-1}\{e^{q_1 u}(1 + q_1\lambda) - e^{q_2 u}(1+q_2\lambda)\}du d\xi, \quad (18)$$

By applying Laplace transform to equation (5), we get the corresponding shear stress as

$$\mu\mathcal{L}\left\{\frac{\partial u}{\partial y}\right\} = \mathcal{L}\left\{\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau\right\}, \quad (19)$$

on simplifications equation (19)

$$\bar{\tau} = \frac{\partial \bar{u}}{\partial y}\left\{\frac{\mu}{(1+q\lambda)}\right\}, \quad (20)$$

In order to find the value of $\frac{\partial \bar{u}}{\partial y}$, we differentiate with respect to y partially equation (16),

$$\frac{\partial \bar{u}}{\partial y} = -\frac{2\, m!\ U}{\pi\ \lambda}\int_0^\infty \cos(y\xi)\frac{1}{q^n}\left\{\frac{(1+\lambda q_1)}{(q_1-q_2)(q-q_1)} - \frac{(1+\lambda q_2)}{(q_1-q_2)(q-q_2)}\right\}d\xi, \quad (21)$$

Employing equation (21) in (20), we obtain

$$\bar{\tau} = \frac{\mu}{(1+\lambda q)}\left[-\frac{2Um!}{\pi\,\lambda}\int_0^\infty \frac{\cos(y\xi)}{q^n}\left\{\frac{(1+\lambda q_1)}{(q-q_1)(q_1-q_2)} - \frac{(1+\lambda q_2)}{(q-q_2)(q_1-q_2)}\right\}d\xi\right], \quad (22)$$

Simplifying equation (22)

$$\bar{\tau} = -\frac{2\ U\ m!\ \mu}{(q_1-q_2)\,\lambda\,\pi}\int_0^\infty \cos(y\xi)\frac{1}{q^m}\left\{\frac{1}{(q-q_1)} - \frac{1}{(q-q_2)}\right\}d\xi, \quad (23)$$

Finally, we apply the inverse Laplace transform to (23), we get the shear stress in the product of convolution

$$\tau = -\frac{2\,\mu\, m\, U\, H(t)}{(q_1-q_2)\,\lambda\,\pi}\int_0^\infty \cos(y\xi)\ t^{m-1} * (e^{q_1 t} - e^{q_2 t})d\xi, \quad (24)$$

Employing the convolution theorem on (24), we find the simple expression in the integral form for shear stress

$$\tau = -\frac{2\,\mu\, m\, U\, H(t)}{(q_1-q_2)\,\lambda\,\pi}\int_0^\infty \int_0^t \cos(y\xi)\,(t-u)^{m-1}(e^{q_1 u} - e^{q_1 u})\, du\, d\xi \quad (25)$$

4. LIMITING CASES

4.1. Newtonian Fluid

Solving and letting $\lambda \to 0$ into equations (18) and (25), we have following mathematical statements

$$\lim_{\lambda\to 0} \lambda(q_1 - q_2) = 1 \,, \quad \lim_{\lambda\to 0} q_1 = -\nu\xi^2 \,, \qquad \lim_{\lambda\to 0} q_2 = -\infty \,,$$

then velocity field and the shear stress for Newtonian fluid are respectively

$$u_N = U\,H(t)t^m - \frac{2\,m\,U\,H(t)}{\pi}\int_0^\infty\int_0^t \frac{\sin(y\xi)}{\xi}\,(t-u)^{m-1}\,e^{-u(\nu\xi^2)}\,du\,d\xi\,, \qquad (26)$$

$$\tau_N = -\frac{2\,\mu\,m\,U\,H(t)}{\pi}\int_0^\infty\int_0^t \cos(y\xi)\,(t-u)^{m-1}\;e^{-u(\nu\xi^2)}\,du\,d\xi. \qquad (27)$$

are acquired.

5. NUMERICAL RESULTS AND DISCUSSION

In this paragraph, the motion of incompressible Maxwell fluid for unidirectional flows is calculated for the solutions of velocity field and shear stress by employing the Fourier Sine and Laplace transforms. The analytical solutions are expressed in the view of convolution product and integral form for the effects of no slip condition and various material parameters which are kinematic viscosity ν, density ρ, viscosity μ, time t, relaxation time λ. These all parameters have special effects on the general solutions of velocity field and shear stress for which behavior of fluid flows has different results. In order to prove similar physical situations and effects of the gained results, limited suitable graphs are sketched in this regard. With typical values numerous diagrams for velocity field and shear stress are depicted. For example, we select $U = 3$, $\rho = 2.413$, and $\nu = 12.46$ for simplicity. By selecting various values for relaxation time λ, the motion of fluid is illustrated for different effects.

6. CONCLUSION

In this paragraph we discuss, the analytical solutions for incompressible Maxwell fluid are calculated for velocity field and shear stress. The integral transforms (the Fourier Sine and Laplace transforms) approach is used. The velocity field and stress are expressed as convolution product and integral form. These general solutions (18) and (25) are reduced for Newtonian fluid when we applied $\lambda \to 0$. Both the solutions for velocity field and shear stress are graphically depicted graphical for the fact of correctness and accuracy with comparisons of known solution of Newtonian fluid. The analysis of this paper has some interesting outcomes which are listed below:

- The general solutions (18) and (25) are particularized for Newtonian fluid when we substituted $\lambda \to 0$.
- The general solutions are particularized and reduced to produce the related solutions for Newtonian fluids.
- The velocity field and shear stress both are increasing functions with regard to distinct values of t. **(See Fig. 1)**
- For different values of ν, the effects of general solutions (18) and (25) are increasing with respect to different behavior of fluid. **(See Fig. 2)**
- The relaxation time λ as conjectured with intersecting effects at the motion of fluid. **(See Fig. 3)**
- An increase of m has the increase profile of velocity field and shear stress. **(See Fig. 4)**
- The depicted Figure 5 has clear comparison in which velocity field is decreasing and shear stress is increasing function with regard to the different values of y.

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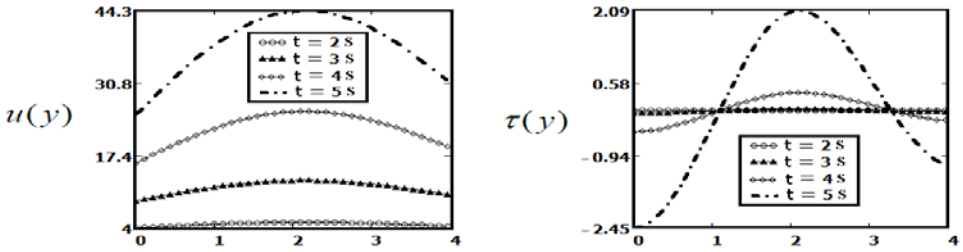


Fig. 1: For $U = 3,\ \nu = 11.72,\ \mu = 6.11,\ \lambda = 2$ and different values of *t* for equations (18) and (25), Profiles of velocity field and shear stress for Maxwell fluid is shown in graphs above.

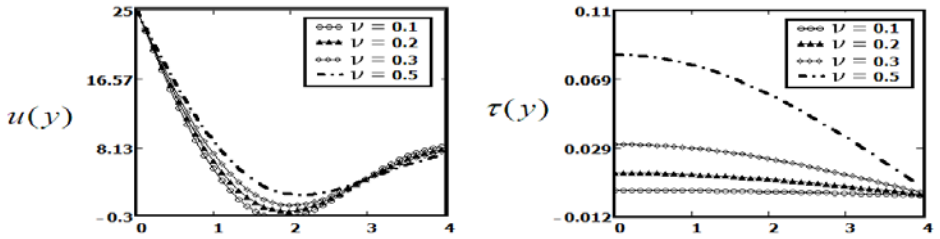


Fig. 2: For $U = 3,\ \rho = 12.051,\ t = 3\ s,\ \lambda = 2$ and different values of ν for equations (18) and (25), Profiles of velocity field and shear stress for Maxwell fluid is shown in graphs above.

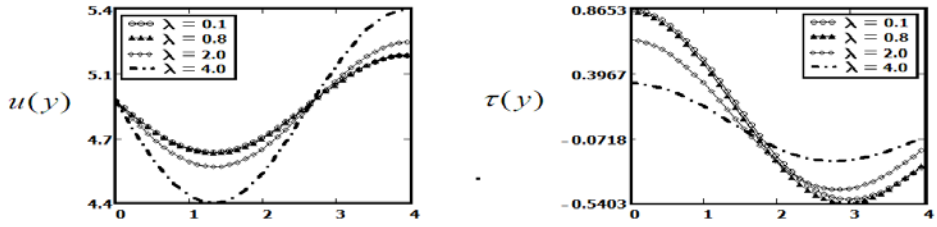


Fig. 3: For $U = 3, \nu = 11.72,\ \mu = 6.11,\ t = 3\ s$ and different values of λ for equations (18) and (25), Profiles of velocity field and shear stress for Maxwell fluid is shown in graphs above.

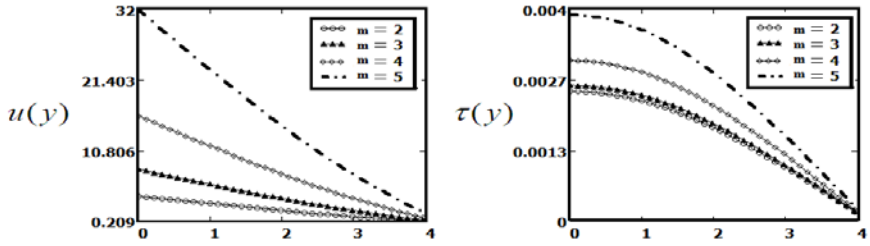


Fig. 4: For $U = 3,\ \nu = 11.72,\ \mu = 6.11,\ \lambda = 2,\ t = 3\ s$ and different values of m for equations (18) and (25), Profiles of velocity field and shear stress for Maxwell fluid is shown in graphs above.

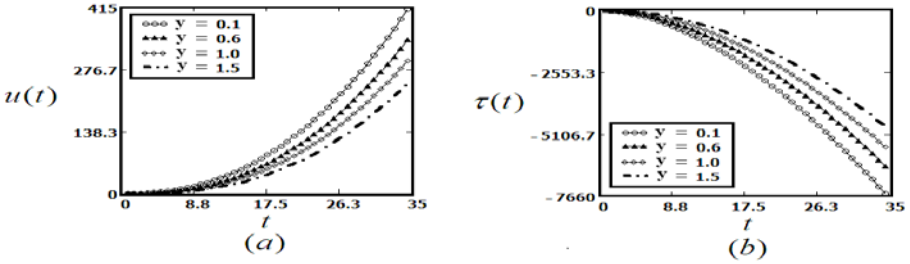


Fig. 5: For $U = 3, \nu = 11.72,\ \mu = 6.11,\ \lambda = 2,\ t = 3\ s$ and different values of y for equations (18) and (25), Profiles of velocity field and shear stress for Maxwell fluid is shown in graphs above.

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