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On Consistency, Stability and Convergence of a Modified Ordinary Differential Equation Solver

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Abstract: In this piece of work, a Modified Ordinary Differential Equation Solver, proposed for the numerical solution of initial value problems in ordinary differential equations is analyzed on the basis of consistency, stability and convergence - characteristics essential for a numerical method to be of any use. A comparison is made between the regions of absolute stability of the Improved Euler method and the Modified Ordinary Differential Equation Solver; the later having the same order of accuracy as the former, possesses a larger stability region

Keywords: Ordinary differential equation, **i**nitial value problem, consistency, stability, convergence.

1. INTRODUCTION

Initial value problems of ordinary differential equations (ODEs) play an important role in engineering and scientific studies. These problems often arise as mathematical models of various physical problems such as population dynamics, radioactive decay, mechanical systems, electrical networks, fluid flows, rate of chemical reactions etc. (Boyce and DiPrima, 2001). In that situation where the analytical solution to the problem under consideration does not exist which is usually the case with nonlinear problem, we resort to numerical techniques to obtain an approximate solution.

Numerical techniques for initial value problems of ODEs have gradually developed as effective mathematical tools. While some researchers have taken efforts to improve the existing conventional or the standard ODE solvers, see for example, (Abraham, 2007 and 2008; Akanbi, 2010; Chandio and Memon, 2010; Qureshi *et al.*, 2013; Qureshi *et al.*, 2014); others have proposed nonstandard ODE solvers based on exponential sums, trigonometric and rational interpolants suitable for the initial value problems having exponential, periodic or singular solution respectively. For a detailed review of such ODE solvers refer, for example, (Glaser and Rokhlin, 2009; Kama and Ibijola, 2001; Ibijola and Sinkala, 2012; Fatunla, 1976; Hellemans, 1982; Ibijola, 1993; Ibijola and Ogunrinde, 2010; Ogunrinde and Fadugba, 2012; Lambert and Shaw, 1965; Luke *et al.*, 1975; Van Niekerk, 1988; Ramos, 2007). The Modified ODE solver developed by Memon *et al*. (2014a) as an improvement on the Improved Euler's Method, is given by the following iterative scheme:

$$\left.\begin{aligned}
&y_n = y_{n-1} + \frac{\Delta t}{2}\left(k_1 + k_2\right)\\
&\text{where}\\
&k_1 = g\left(t_{n-1}, y_{n-1}\right)\\
&\tilde{k} = g\left(t_{n-1} + \frac{\Delta t}{2}, y_{n-1} + \frac{\Delta t}{2}k_1\right)\\
&k_2 = g\left(t_{n-1} + \Delta t, y_{n-1} + \Delta t\,\tilde{k}\right)
\end{aligned}\right\} \qquad (1)$$

In the present paper, the ODE Solver given by Eq. (1) is investigated for consistency, stability and convergence – characteristics that play an essential role in the overall efficiency of a numerical scheme.

2. MATERIALS AND METHODS

A single-step method can be written in the form:

$$y_n = y_{n-1} + \Delta t\,\phi\left(t, y; \Delta t\right) \qquad (2)$$

where $\phi\left(t, y; \Delta t\right)$ is known as the increment function of the method.

2.1 CONSISTENCY

Definition 01 Given an initial value problem $y' = g\left(t, y\right); y\left(t_0\right) = y_0$, a numerical method with an increment function $\phi\left(t, y; \Delta t\right)$ is said to be consistent if

$$\lim_{\Delta t \to 0} \phi\left(t, y; \Delta t\right) = g\left(t, y\right) \qquad (3)$$

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The increment function of the modified ODE solver is

$$\phi(t_{n-1}, y_{n-1}; \Delta t) = \frac{1}{2}(k_1 + k_2)$$

$$\therefore \lim_{\Delta t \to 0} \phi(t_{n-1}, y_{n-1}; \Delta t) = \frac{1}{2} \lim_{\Delta t \to 0} (k_1 + k_2)$$

$$= \frac{1}{2} \lim_{\Delta t \to 0} \left(g(t_{n-1}, y_{n-1}) + g \left(\begin{array}{l} t_{n-1} + \Delta t, y_{n-1} + \\ \Delta t\, g\left(t_{n-1} + \frac{\Delta t}{2}, y_{n-1} + \frac{\Delta t}{2} g(t_{n-1}, y_{n-1}) \right) \end{array} \right) \right)$$

$$= g(t_{n-1}, y_{n-1})$$

Hence, the modified ODE solver is consistent.

Theorem 01 Let $c_0, c_1, c_2, \ldots, c_n$ be real numbers satisfying

$$|c_{i+1}| \le (1+a)|c_i| + b, \quad a > 0,\ b \ge 0,\ i = 0,1,2,\ldots,n-1$$

then $\quad |c_n| \le e^{na} |c_0| + \dfrac{e^{na} - 1}{a} b$

Proof. From the assumptions we get immediately

$$|c_1| \le (1+a)|c_0| + b$$

$$|c_2| \le (1+a)|c_1| + b$$

$$= (1+a)^2 |c_0| + (1+a)b + b$$

$$|c_3| \le (1+a)|c_2| + b$$

$$= (1+a)^3 |c_0| + (1+a)^2 b + (1+a)b + b \vdots$$

$$|c_n| \le (1+a)^n |c_0| + b\left[1 + (1+a) + (1+a)^2 + \cdots + (1+a)^{n-1}\right]$$

$$= (1+a)^n |c_0| + b \frac{(1+a)^n - 1}{a}$$

$$\le e^{na} |c_0| + b \frac{e^{na} - 1}{a}$$

Since $0 < 1 + a \le e^a$ for $a > -1$.

Theorem 02 Let (t_{n-1}, y_{n-1}) and $(t_{n-1}, \hat{y}_{n-1})$ be any two points in the region *D* defined by $D = \left\{ (t, y) \in R^2 \,\middle|\, t_0 \le t \le t_n, -\infty < y < \infty \right\}$, and that g is a Lipschitz continuous function on *D* such that

$$|g(t_{n-1}, y_{n-1}) - g(t_{n-1}, \hat{y}_{n-1})| \le L|y_{n-1} - \hat{y}_{n-1}|,$$

then the increment function ϕ is Lipschitz continuous, and

$$|\phi(t_{n-1}, y_{n-1}; \Delta t) - \phi(t_{n-1}, \hat{y}_{n-1}; \Delta t)| \le \hat{L} |y_{n-1} - \hat{y}_{n-1}|$$

where L and $\hat{L}$ are, respectively, the Lipschitz constants for g and ϕ.

Proof. From the assumptions, we have

$$|k_1 - \hat{k}_1| = |g(t_{n-1}, y_{n-1}) - g(t_{n-1}, \hat{y}_{n-1})| \le L|y_{n-1} - \hat{y}_{n-1}|$$

$$|\tilde{k} - \hat{\tilde{k}}| = \left| g\left(t_{n-1} + \frac{\Delta t}{2}, y_{n-1} + \frac{\Delta t}{2} k_1\right) - g\left(t_{n-1} + \frac{\Delta t}{2}, \hat{y}_{n-1} + \frac{\Delta t}{2} \hat{k}_1\right) \right|$$

$$\le L \left| \left(y_{n-1} + \frac{\Delta t}{2} k_1 \right) - \left(\hat{y}_{n-1} + \frac{\Delta t}{2} \hat{k}_1 \right) \right|$$

$$= L \left| (y_{n-1} - \hat{y}_{n-1}) + \frac{\Delta t}{2} (k_1 - \hat{k}_1) \right|$$

$$\le L |(y_{n-1} - \hat{y}_{n-1})| + \frac{1}{2} L \Delta t |k_1 - \hat{k}_1|$$

$$\le L |(y_{n-1} - \hat{y}_{n-1})| + \frac{1}{2} L^2 \Delta t |y_{n-1} - \hat{y}_{n-1}|$$

$$= \left(L + \frac{1}{2} L^2 \Delta t \right) |y_{n-1} - \hat{y}_{n-1}|$$

$$|k_2 - \hat{k}_2| = \left| g\left(t_{n-1} + \Delta t, \mathrm{y}_{n-1} + \Delta t \tilde{k}\right) - g\left(t_{n-1} + \Delta t, \hat{\mathrm{y}}_{n-1} + \Delta t \hat{\tilde{k}}\right) \right|$$

$$\le L \left| \left(y_{n-1} + \Delta t \tilde{k} \right) - \left(\hat{y}_{n-1} + \Delta t \hat{\tilde{k}} \right) \right|$$

$$\le L |y_{n-1} - \hat{y}_{n-1}| + L \Delta t \left| \tilde{k} - \hat{\tilde{k}} \right|$$

$$\le L |y_{n-1} - \hat{y}_{n-1}| + L \Delta t \left(L + \frac{1}{2} L^2 \Delta t \right) |y_{n-1} - \hat{y}_{n-1}|$$

$$= \left(L + L^2 \Delta t + \frac{1}{2} L^3 (\Delta t)^2 \right) |y_{n-1} - \hat{y}_{n-1}|$$

Hence, for the increment function ϕ of the Modified ODE Solver:

$$|\phi(t, y; \Delta t) - \phi(t, \hat{y}; \Delta t)|$$

$$= \frac{1}{2} \left| (k_1 + k_2) - (\hat{k}_1 + \hat{k}_2) \right|$$

$$\le \frac{1}{2} |k_1 - \hat{k}_1| + \frac{1}{2} |k_2 - \hat{k}_2|$$

$$\le \frac{L}{2} |y_{n-1} - \hat{y}_{n-1}| + \frac{1}{2} \left(L + L^2 \Delta t + \frac{1}{2} L^3 (\Delta t)^2 \right) |y_{n-1} - \hat{y}_{n-1}|$$

$$= \hat{L} \, |y_{n-1} - \hat{y}_{n-1}|$$

where $\hat{L} = L + \frac{1}{2}L^2\Delta t + \frac{1}{4}L^3(\Delta t)^2$

Thus, the increment function of the Modified ODE Solver is Lipschitz continuous.

2.2 STABILITY

Theorem 03 Let y_n and $\hat{y}_n$ be two solutions to the differential equation $y' = g(t, y)$, generated by a numerical method, subject to the initial conditions $y(t_0) = y_0$ and, $\hat{y}(t_0) = \hat{y}_0$ respectively, such that $|y_0 - \hat{y}_0| < \varepsilon,\ \varepsilon > 0$. The condition

$$|y_n - \hat{y}_n| \le K\,|y_0 - \hat{y}_0|\,, \qquad K > 0$$

is the necessary and sufficient condition for the numerical method to be stable.

Applying theorem 03 to the modified ODE solver gives,

$$y_n = y_{n-1} + \Delta t\ \phi\left(t_{n-1}, y_{n-1}; \Delta t\right) \qquad (4)$$

and $$\hat{y}_n = \hat{y}_{n-1} + \Delta t\ \phi\left(t_{n-1}, \hat{y}_{n-1}; \Delta t\right) \qquad (5)$$

This implies that

$$|y_n - \hat{y}_n| \le |y_{n-1} - \hat{y}_{n-1}| + \Delta t\left|\phi(t_{n-1}, y_{n-1}; \Delta t) - \phi(t_{n-1}, \hat{y}_{n-1}; \Delta t)\right|$$

$$\le |y_{n-1} - \hat{y}_{n-1}| + \hat{L}\,\Delta t\ |y_{n-1} - \hat{y}_{n-1}|$$

$$= \left(1 + \hat{L}\,\Delta t\right)|y_{n-1} - \hat{y}_{n-1}|$$

$$= \left(1 + \hat{L}\,\Delta t\right)^n |y_0 - \hat{y}_0|$$

Using theorem 01 with $a = \hat{L}\,\Delta t$ and $b = 0$ gives,

$$|y_n - \hat{y}_n| \le K\,|y_0 - \hat{y}_0| \qquad , \qquad K = e^{\hat{L}\,\Delta t\, n}$$

Hence, the modified ODE solver is stable.

2.3 ABSOLUTE STABILITY

Applying the Modified ODE Solver to Dahlquist's model problem (1963) gives,

$$k_1 = \lambda\, y_{n-1} \qquad \tilde{k} = \lambda\left(1 + \frac{1}{2}\lambda\,\Delta t\right)y_{n-1}$$

$$k_2 = \lambda\left(1 + \lambda\,\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2\right)y_{n-1}$$

Then, $$y_n = y_{n-1} + \frac{\Delta t}{2}\left(k_1 + k_2\right)$$

$$= \left\{1 + \lambda\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2 + \frac{1}{4}\lambda^3\left(\Delta t\right)^3\right\}y_{n-1}$$

$$\therefore\ y_n = \left\{1 + \lambda\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2 + \frac{1}{4}\lambda^3\left(\Delta t\right)^3\right\}^n y_0$$

$$= \left\{1 + \lambda\,\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2 + \frac{1}{4}\lambda^3\left(\Delta t\right)^3\right\}^n \quad \because y_0 = y(0) = 1$$

$$= \left\{1 + \lambda\,\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2 + \frac{1}{4}\lambda^3\left(\Delta t\right)^3\right\}^{\left(\frac{t_n}{\Delta t}\right)}$$

$$\ln y_n = \frac{t_n}{\Delta t}\ln\left\{1 + \lambda\,\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2 + \frac{1}{4}\lambda^3\left(\Delta t\right)^3\right\}$$

$$\lim_{\Delta t \to 0}\ln y_n = t_n \lim_{\Delta t \to 0}\frac{1}{\Delta t}\ln\left\{\begin{aligned}&1 + \lambda\,\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2\\ &+ \frac{1}{4}\lambda^3\left(\Delta t\right)^3\end{aligned}\right\}$$

By applying l'Hopital's rule, we get

$$\lim_{\Delta t \to 0}\ln y_n = \lambda\, t_n,$$

which implies that $\ln \lim_{\Delta t \to 0} y_n = \lambda\, t_n$, and thus,

$$\lim_{\Delta t \to 0} y_n = e^{\lambda t_n}.$$

Hence, the approximate solution y_n approaches the analytical solution of the model problem as $\Delta t \to 0$.

The ratio y_n / y_{n-1} is known as stability function, $R(z)$, hence, the stability function $R(z)$ of the Modified ODE Solver is given by

$$R(z) = 1 + \lambda\,\Delta t + \frac{1}{2}\lambda^2\left(\Delta t\right)^2 + \frac{1}{4}\lambda^3\left(\Delta t\right)^3$$

Let $z = \lambda\,\Delta t$, the stability function of the Modified ODE Solver is obtained as

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3 \qquad (6)$$

While that for the Improved Euler method is given as:

$$R(z) = 1 + z + \frac{1}{2}z^2 \qquad (7)$$

Stability region is given by

$$\left\{z \in \mathbb{C} : \left|R(z)\right| \le 1\right\} \qquad (8)$$

as shown in figures 01 and 02 where the shaded parts denote the stability regions for Improved Euler method and the Modified ODE Solver respectively.

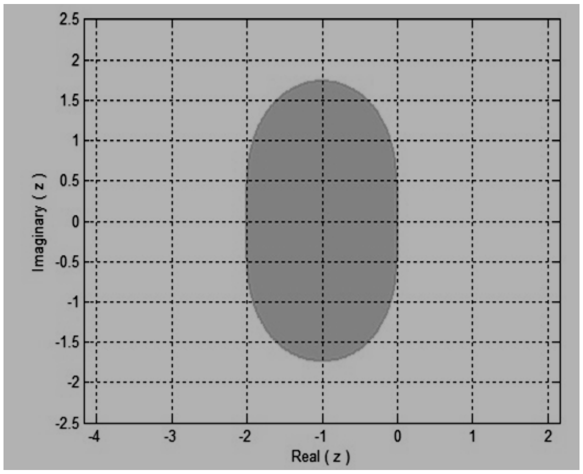

Fig. 1. Stability region for Improved Euler method

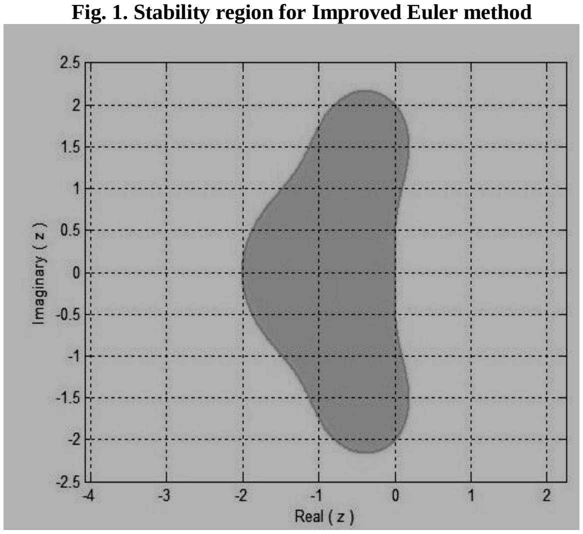

Fig. 2. Stability region for Modified ODE Solver

2.4 CONVERGENCE

Definition 02 Let $y(t_r)$ be the exact solution of the initial value problem $y' = g(t, y); y(t_0) = y_0$ at $t = t_r$ and y_r be the corresponding approximate solution generated by a numerical method. The global truncation error or global error e_r at $t = t_r$ is given as

$$e_r = y(t_r) - y_r$$

It is to be noted that since $y_0 = y(t_0)$, $e_0 = 0$.

Definition 03 A numerical method is convergent if for every initial value problem $y' = g(t, y); y(t_0) = y_0$ with a Lipschitz function g and every $t > 0$

$$\lim_{\Delta t \to 0} \max_{0 \le r \le N} \| y(t_r) - y_r \| = 0,$$

that is, the numerical solution approaches the true solution as the step size Δt tends to zero.

Consider the Modified ODE solver:

$$y_{i+1} - y_i - \Delta t\, \phi(t_i, y_i; \Delta t) = 0 \qquad (9)$$

The quantity

$$\tau_i = y(t_{i+1}) - y(t_i) - \Delta t\, \phi(t_i, y(t_i); \Delta t) \qquad (10)$$

is referred to as the local error of the Modified ODE solver. It provides a measure of the extent to which the exact solution fails to satisfy the difference equation of the Modified ODE solver. On subtracting Eq. (9) from Eq. (10), it is deduced that

$$e_{i+1} = e_i + \Delta t \left[\phi(t_i, y(t_i); \Delta t) - \phi(t_i, y_i; \Delta t) \right] + \tau_i$$

$$\Rightarrow |e_{i+1}| \le |e_i| + \Delta t \left| \phi(t_i, y(t_i); \Delta t) - \phi(t_i, y_i; \Delta t) \right| + |\tau_i|$$

Using theorems 01 and 02,

$$|e_{i+1}| \le \left(1 + \Delta t \hat{L}\right) |e_i| + |\tau_i|$$

Let $T = \max|\tau_i|, i = 0, 1, 2, \ldots, k-1$, then using theorem 01,

$$|e_r| \le e^{r \Delta t \hat{L}} |e_0| + \frac{T}{\Delta t \hat{L}} \left[e^{r \Delta t \hat{L}} - 1 \right]$$

Since $e_0 = 0$ and $\Delta t = (t_r - t_0)/r$, the global error bound for the Modified ODE Solver is obtained as

$$|e_r| \le \frac{T}{\Delta t\, \hat{L}} \left[e^{\hat{L}(t_r - t_0)} - 1 \right],$$

T being an upper bound on its local error.

$$\because T = C(\Delta t)^3,\ C = \frac{1}{2} M N^2$$

as given in Memon *et al*. (2014b). Hence, the global error bound for the Modified ODE Solver is given as

$$|e_r| \le C(\Delta t)^2 \left[\frac{e^{\hat{L}(t_r - t_0)} - 1}{\hat{L}} \right],\ C = \frac{1}{2} M N^2 \qquad (11)$$

$$\therefore \lim_{\Delta t \to 0} |e_r| = 0$$

$$\Rightarrow \lim_{\Delta t \to 0} |y(t_r) - y_r| = 0$$

This shows that the approximate solution approaches the exact solution with increasingly small step size Δt. Thus, the Modified ODE Solver is convergent.

3. CONCLUSION

It is concluded that the modified ODE solver is consistent, stable and convergent. Further, it possesses a larger stability region in comparison to the Improved Euler method, thus enabling the modified method to work at relatively larger step size while maintaining both accuracy and stability.

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