Sindh Univ. Res. Jour. (Sci. Ser.) Vol. 47 (4) 649-652 (2015)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

Common Tripled Fixed Point Theorem in Dislocated Quasi-Metric Space

M.U. RAHMAN[++], M. SARWAR, P. S. KUMARI*

Department of Mathematics, University of Malakand, Chakdara Dir(L), Pakistan

Received 12th October 2014 and Revised 21st August 2015

Abstract: The purpose of this note is to establish a tripled fixed point theorem in the frame work of dislocated quasi-metric space, which extend and generalize similar type of fixed point results. In the support of the establish theorem an example is given.
2010 Mathematics Subject Classification: 47H10, 54H25.

Keywords: Complete dislocated quasi-metric space, self-mapping, Cauchy sequence, tripled coincidence and tripled fixed point, W - compatible map- pings.

1. INTRODUCTION

The idea of dislocated metric (d-metric) space was given by Hitzler and Seda (Hitzler and Seda, 2000). In this type of space self-distance between points need not to be zero necessarily. They also established well-known Banach contraction theorem in dislocated metric space. d-metric space play a vital role in Topology, Logical Programming and Electronic Engineering etc. In 2005, (Zeyada, *et al.,* 2005) initiated the notion of complete dislocated quasi-metric (dq-metric) space and proved the result of Hitzler and Seda (Hitzler and Seda, 2000) in dq-metric space. With the passage of time many papers have been published containing fixed point theorems for a single and a pair of mappings for different type of contractive conditions in dq-metric space (see Aage. and Salunke, 2008), (Sarwar, *et al.,* 2014) (Rahman and Sarwar 2014), (Rahman and Sarwar, 2014).

Bhaskar and Lakshmikantham initiated the idea of coupled fixed point for non-linear contractions in partially ordered metric spaces. Furthermore, after Bhaskar and Lakshmikantham (Bhaskar and Lakshmikantham, 2006) coupled fixed point theorems become a dominant research topic (see Akcay and Alaca, 2012), (Kumar and Vashistha, (2013). Recently, a fixed point of order $N \geq 3$ was established by samet and vetro.

In this article we have proved a common tripled fixed point theorem in dq-metric space, which extend and generalize similar type of comparable fixed point results in the literature.

2. PRELIMINARIES

Definition 2.1. (Zeyada, *et al.,* 2005) Suppose X be a non-empty set and $d_1: A \times A \to R^+$ be a function satisfying the conditions

1) $d_1(a, b) = d_1(b, a) = 0$ implies $a = b$;
2) $d_1(a, b) \leq d_1(a, c) + d_1(c, b)$ for all $a, b, c \in A$.

Then d_1 is called dq-metric and the pair (A, d_1) is called dq-metric space.

Every metric and dislocated metric spaces are dislocated quasi-metric spaces but the converse is not true as clear from the following example.

Example 2.1. Let $A = R$ define the distance function $d_1 : A \times A \to R^+$ by $d_1(a, b) = |a|$ for all $a, b \in A$.

The following definitions can be seen in (Zeyada, *et al.,* 2005).

Definition 2.2. A sequence $\{l_n\}$ is called dislocated quasi-convergent (dq-convergent) in A if for $n \in N$ we have

$$\lim d_1(l_n, l) = \lim d_1(l, l_n) = 0.$$

In this case l is called dislocated quasi-limit (dq-limit) of the sequence $\{l_n\}$.

Definition 2.3. A sequence $\{l_n\}$ in dq-metric space is called Cauchy sequence if for $\epsilon > 0$ there exists a positive integer n_0 such that for $m, n \geq n_0$, we have $d_1(l_m, l_n) < \epsilon$.

Definition 2.4. A dq-metric space (A, d_1) is said to be complete if every Cauchy sequence in A converges to a point in A.

The following famous results can be found in (Zeyada, *et al.,* 2005).

Lemma 1. Limit of convergent sequence in dq-metric space is single one.

Theorem 2.1. Suppose (A, d_1) be a complete dq-metric space and $J: A \to A$ be a contraction.

[++]Corresponding author: mujeeb846@yahoo.com sarwar@uom.edu.pk
*KL University, Green Fields, Vaddeswaram Guntur District, Andhra Paradesh 522502, India.

Then J has a single fixed point.

Definition 2.5. (Bhaskar and Lakshmikantham, 2006). An element $(a, b) \in A^2$ is called coupled fixed point of the mapping $J : A \times A \to A$ if $J(a, b) = a$ and $J(b, a) = b$ for $a, b \in A$.

Example 2.2. Suppose $A = R$ and $J : A \times A \to A$ defined by

$$J(a, b) = ab/2.$$

Here (0, 0) is the coupled fixed point of J.

Definition 2.6. (Ayadi and Abbas, (2012) A point $(a\ b, c) \in A^3$ is called a tripled fixed point of the mapping

$$J : A^3 \to A$$

if

$J(a, b, c) = a$, $J(b, c, a) = b$, $J(c, a, b) = c$ $\forall a, b, c \in A$.

Example 2.3. Let $A = R$ and $J : A^3 \to A$ defined by

$$J(a, b, c) = abc/6.$$

Here (0, 0, 0) is a tripled fixed point of J.

Definition 2.7. (Ayadi and Abbas, (2012) A point $(a, b, c) \in A^3$ is called a tripled coincidence point of the mappings $J : A^3 \to A$ and $K : A \to A$ if $J(a, b, c) = Ka$, $J(b, c, a) = Jb$, $J(c, a, b) = Jc$ $\forall a, b, c \in A$, where (Ka, Kb, Kc) is called tripled point of coincidence of the mappings J and K.

Definition 2.8. (Ayadi and Abbas, (2012) A point $(a, b, c) \in A^3$ is called common tripled coincidence point of the mappings $J: A^3 \to A$ and $K: A \to A$ if $J(a, b, c) = Ka = a$, $J(b, c, a) = Kb = b$ and $J(c, a, b) = Kc = c$ $\forall a, b, c \in A$.

Definition 2.9. (Ayadi and Abbas, (2012) Mappings $J : A^3 \to A$ and $K : A \to A$ are said to be W-compatible if $KJ(a, b, c) = J(Ka, Kb, Kc)$.
Whenever $J(a, b, c) = Ka$, $J(b, c, a) = Kb$ and $J(c, a, b) = Kc$ $\forall a, b, c \in A$.

Definition 2.10. Mappings $J : A^3 \to A$ and $K : A \to A$ are said to be commutative mappings if $KJ(a, b, c) = J(Ka, Kb, Kc)$.

3. **MAIN RESULTS**

Theorem 3.1. Suppose (A, D) be a dq-metric space. $G : A^3 \to A$ and $h : A \to A$ be continuous and commutative mappings satisfying

(1) $G(A^3) \subseteq h(A)$;
(2) $h(A)$ is complete subspace of A;
(3) $D(G(a, b, c), G(j, k, l)) \le \alpha \cdot D(ha, hj) + \beta \cdot d(hb, hk) + \gamma \cdot D(hc, hl)$ for all $a, b, c, j, k, l \in A$ and α, β and γ are non-negative constants with $\alpha+\beta+\gamma < 1$.

Then G and h have tripled coincidence point. Moreover, if G and h are W-compatible, then G and h have a unique tripled common fixed point which is in the form (j, j, j) for some j in A.

Proof. Using (1) we construct three sequences $\{ha_n\}$, $\{hb_n\}$ and $\{hc_n\}$ by the following rule
$ha_{n+1} = G(a_n, b_n, c_n)$, $hb_{n+1} = G(b_n, c_n, a_n)$ and $hc_{n+1} = G(c_n, a_n, b_n)$.
Using (3) and the above construction we have
$D(ha_n, ha_{n+1}) = D(G(a_{n-1}, b_{n-1}, c_{n-1}), G(a_n, b_n, c_n))$
$D(ha_n, ha_{n+1}) \le \alpha \cdot D(ha_{n-1}, ha_n) + \beta \cdot D(hb_{n-1}, hb_n) + \gamma \cdot d(hc_{n-1}, hc_n)$. (1)
Similarly, we can show that
$D(hb_n, hb_{n+1}) \le \alpha \cdot D(hb_{n-1}, hb_n) + \beta \cdot D(hc_{n-1}, hc_n) + \gamma \cdot D(ha_{n-1}, ha_n)$ (2)
$D(hc_n, hc_{n+1}) \le \alpha \cdot D(hc_{n-1}, hc_n) + \beta \cdot D(ha_{n-1}, ha_n) + \gamma \cdot D(hb_{n-1}, hb_n)$. (3)
Adding Eq.(1), Eq.(2) and Eq.(3) we have
$[D(ha_n, ha_{n+1}) + D(hb_n, hb_{n+1}) + D(hc_n, hc_{n+1})] \le (\alpha + \beta + \gamma) \cdot [D(ha_{n-1}, ha_n) + D(hb_{n-1}, hb_n) + D(hc_{n-1}, hc_n)]$.
Since $\alpha + \beta + \gamma < 1$, so assume $\alpha + \beta + \gamma = r < 1$, thus the above inequality take the form

Also, we have
$[D(ha_n, ha_{n+1}) + D(hb_n, hb_{n+1}) + D(hc_n, hc_{n+1})]$
$\le r \cdot [D(ha_{n-1}, ha_n) + D(hb_{n-1}, hb_n) + D(hc_{n-1}, hc_n)]$.
$[D(ha_n, ha_{n+1}) + D(hb_n, hb_{n+1}) + D(hc_n, hc_{n+1})] \le r^2 \cdot [D(ha_{n-2}, ha_{n-1}) + D(hb_{n-2}, hb_{n-1}) + D(hc_{n-2}, hc_{n-1})]$.
Continuing the same procedure we have
$[D(ha_n, ha_{n+1}) + D(hb_n, hb_{n+1}) + D(hc_n, hc_{n+1})] \le r^n \cdot [D(ha_0, ha_1) + D(hb_0, hb_1) + D(hc_0, hc_1)]$.
Taking limit $n \to \infty$, we have
$\lim [D(ha_n, ha_{n+1}) + D(hb_n, hb_{n+1}) + D(hc_n, hc_{n+1})] = 0$.
Implies that

$\lim D(ha_n, ha_{n+1}) = 0$, $\lim D(hb_n, hb_{n+1}) = 0$ and $\lim D(hc_n, hc_{n+1})] = 0$.
Thus $\{ha_n\}$, $\{hb_n\}$ and $\{hc_n\}$ are Cauchy sequences in complete subspace (h(A), D), so there must exists $a, b, c \in A$ such that
$ha_n \to a$, $hb_n \to b$ and $hc_n \to c$ as $n \to \infty$.
Using the continuity and commutatively of G and h, we have
$ha_{n+1} = G(a_n, b_n, c_n) \Rightarrow hha_{n+1} = hG(a_n, b_n,$

$c_n) = G(ha_n, hb_n, hc_n)$.

Taking limit $n \to \infty$, we get

$$ha = G(a, b, c).$$

Using similar procedure we have

$$hb = G(b, c, a) \text{ and } hc = G(c, a, b).$$

Hence $(a, b, c) \in A^3$ is a common tripled coincidence point of G and h, while (ha, hb, hc) is called tripled point of coincidence.

Next to show that tripled point of coincidence is single for this assume that (ha, hb, hc) and (ha_1, hb_1, hc_1) are two distinct tripled point of coincidence. i.e

$ha = G(a, b, c)$

$hb = G(b, c, a)$

$hc = G(c, a, b)$

and

$ha_1 = G(a_1, b_1, c_1)$

$hb_1 = G(b_1, c_1, a_1)$

$hc_1 = G(c_1, a_1, b_1)$.

Now using (3), we have

$D(ha, ha_1) = D(G(a, b, c), G(a_1, b_1, c_1))$

$D(ha, ha_1) \le \alpha \cdot D(ha, ha_1) + \beta \cdot D(hb, hb_1) + \gamma \cdot D(hc, hc_1)$. (4)

Similarly, we can show that

$D(hb, hb_1) \le \alpha \cdot D(hb, hb_1) + \beta \cdot D(hc, hc_1) + \gamma \cdot D(ha, ha_1)$ (5)

$D(hc, hc_1) \le \alpha \cdot D(hc, hc_1) + \beta \cdot D(ha, ha_1) + \gamma \cdot D(hb, hb_1)$. (6)

Adding Eq.(4), Eq.(5) and Eq.(6) we have

$[D(ha, ha_1)+D(hb, hb_1)+D(hc, hc_1)] \le (\alpha+\beta+\gamma)\cdot[D(ha, ha_1)+D(hb, hb_1)+D(hc, hc_1)]$.

The above inequality is possible only if

$[D(ha, ha_1) + D(hb, hb_1) + D(hc, hc_1)] = 0$

which implies that

$D(ha, ha_1) = D(hb, hc_1) = D(hc, hc_1) = 0$.

Similarly, we can show that

$D(ha_1, ha) = D(hb_1, hb) = D(hc_1, hc) = 0$.

Implies that

$D(ha_1, ha) = D(ha, ha_1)$, $D(hb_1, hb) = D(hb, hb_1)$ and $D(hc_1, hc) = D(hc, hc_1)$.

Hence

$ha = ha_1$, $hb = hb_1$ and $hc = hc_1$. (7)

Thus tripled point of coincidence is single. i.e

$(ha, hb, hc) = (ha_1, hb_1, hc_1)$.

Next we have to show that $ha = hb = hc$ for this consider

$D(ha, hb_1) = D(G(a, b, c), G(b_1, c_1, a_1))$

$D(ha, hb_1) \le \alpha \cdot D(ha, hb_1) + \beta \cdot d(hb, hc_1) + \gamma \cdot D(hc, ha_1)$. (8)

Similarly, we can show that

$D(hb, hc_1) \le \alpha \cdot D(hb, hc_1) + \beta\ D(hc, ha_1) + \gamma\cdot D(ha, hb_1)$ (9)

$D(hc, ha_1) \le \alpha \cdot D(hc, ha_1) + \beta \cdot D(ha, hb_1) + \gamma \cdot D(hb, hc_1)$. (10)

Adding Eq.(8), Eq.(9) and Eq.(10) we have

$[D(ha, hb_1)+D(hb, hc_1)+D(hc, ha_1)] \le (\alpha+\beta+\gamma)\cdot[D(ha, hb_1)+D(hb, hc_1)+D(hc, ha_1)]$.

Since $\alpha + \beta + \gamma < 1$, so the above inequality is possible only if

$[D(ha, hb_1) + D(hb, hc_1) + D(hc, ha_1)] = 0$.

Implies that

$D(ha, hb_1) = D(hb, hc_1) = D(hc, ha_1) = 0$.

Similarly, we can show that

$D(hb_1, ha) = D(hc_1, hb) = D(ha_1, hc) = 0$.

Hence

$ha = hb_1$, $hb = hc_1$ and $hc = ha_1$. (11)

From Eq.(7) and Eq.(11) we have

$ha = hb = hc$. (12)

Let $u = gx$ then by Eq.(12) we get

$p = ha = G(a, b, c) = hb = G(b, c, a) = hc = G(c, a, b)$. (13)

Also, since G and h are W-compatible so

$G(ha, hb, hc) = hG(a, b, c)$.

Using Eq.(13) we have

$G(p, p, p) = hp$.

Thus (p, p, p) become a tripled coincidence point of G and h, while (hp, hp, hp) is tripled point of coincidence. Since point of coincidence is unique so

$(ha, ha, ha) = (hp, hp, hp) = ha = hp = p = G(p, p, p)$.

Therefore (p, p, p) is a single common tripled fixed point of G and h.

One can deduced several corollaries from the above theorem.

Now we present an example to illustrate our main result.

Example 3.1. Suppose $A = R$, with dq-metric is given by

$D(a, b) = |a|$ for all $a, b \in A$.

Define $G : A^3 \to A$ and $h : A \to A$ by

$G(a, b, c) = a + b + c/12$ and $ha = a/2$

Clearly G and h are continuous mappings and $G(A^3) \subseteq h(A)$, also we see that G and h are commutative mappings. Moreover

$$D(G(a,b,c), G(j,k,l)) = |a+b+c/12| = 1/6|a+b+c/2| \le 1/6(|a/2|+|b/2|+|c/2|) = 1/6|a/2|+1/6|b/2|+1/6|c/2|$$
$$= \alpha \cdot D(ha, hj) + \beta \cdot D(hb, hk) + \gamma \cdot D(hc, hl).$$

So (3) is satisfied for $\alpha = \beta = \gamma = 1/6$. 6

Consequently (a, b, c) is a tripled coincidence point of G and h iff $a = b = c = 0$, which implies that G and h are W-compatible. So by applying our main theorem we get the existence and the uniqueness of tripled common fixed point of G and h which is $(0, 0, 0)$.

REFERENCES:

Aage C. T. and J. N. Salunke, (2008), Some results of fixed point Theorem in dislocated quasi-metric space, Bulletin of Marathadawa Mathematical Society, 9, 1-5.

Akcay D. and C. Alaca, (2012), Coupled fixed point theorem in dislocated quasi-metric space, Journal of Advance Studies in Topology, 4, 66-72.

Ayadi H. and M. Abbas, (2012) Tripled coincidence and fixed point results in partial metric space, Applied and General Topology, 13, 193-206.

Bhaskar T. G. and V. Lakshmikantham, (2006) Fixed point theorem in partially ordered metric spaces and applications, Non-linear Analysis Theorey and Applications, 65, 1379-1393.

Hitzler P. and A. K. Seda, (2000) Dislocated Topologies, Journal of Electrical Engineering, vol. 51, 3-7.

Kumar J. and S. Vashistha, (2013) Coupled fixed point theorem for generalized contraction in complex valued metric spaces, International Journal of Computer Applications, vol. 83, 36-40.

Rahman M. U. and S. Muhammad (2014), Fixed point results in dislocated quasi-metric space, International Mathematical Forum, vol. 9, 677-682.

Rahman M. U. and M. Sarwar, (2014), Fixed point results for some new type of contraction conditions in dislocated quasi-metric space, International Journal of Mathematics and Scientific Computting, vol. 4, 68-71.

Sarwar, M., M. U. Rahman and G. Ali, (2014) Some fixed point results in dislocated quasi-metric (dq-metric) spaces, Journal of Inequalities and Applications, 278, 1-11.

Samet B. and C. Vetro, (2010) Coupled fixed point, f-invarient set and fixed point of N-order, Ann. Funct. Anal., 1, 46-56.

Zeyada, F. M., G. H. Hassan and M. A. Ahmad, (2005) A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space, Arabian J. Sci. Engg., 31, 111-114.