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The Applications of Natural Transform to the Analytical Solutions of Some Fractional Order Ordinary Differential Equations

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Abstract: We introduce the applications of Natural transform method to find the analytical Solutions of some fractional order differential equations and also we introduce basic con- cepts of fractional calculus which is new approach to the analytical solution in this area. First we develop the general setting,then we turn to the applications of Natural transform method (NTM) to some interesting example of nonhomogeneous differential equations of arbitrary order.

Keywords: Fractional calculus, Fractional differential equations; Analytical solutions, Ho-mogeneous and non homogeneous Differential equations, Natural transform method.

1. INTRODUCTION

There are various integral transforms (Khan and Khan. 2008) (Belgacem and Silambarasan, 2012) (Asiru. 2002) (Spiegel.1965) (Watugala. 1998) available to solve differential equations and integral equations, among these transform Laplace trasform is widely used. The Samudu transform was first introduced by Watugala in (1993) which he used in control engineering problems. Later Samudu transform was extended to two variabls (Belgacem, *et al.*, 2003), (Watugala, 2003), by the same author. This transform has many applications in the solutions of differential equations of arbitrary order as well as in the solutions of many engineering problems. Natural transform was established by Khan and Khan (2008) which was first applied to the solution of fluid flow problem then it was applied to the solution of Maxwell's equations (Khan and Khan. 2008) Later on this transform were used in the solution of many ordinary differential equations with integer order (Belgacem and Silambarasan, (2012) (Loonker, and Banerji, 2013). (Loonker, and Banerji, 2013). The speciality of Natural transform is that it converges to both Laplace transform and Samudu transform just only by changing the variables. It works as a check on these two mentioned transforms. The applications of Natural Transform method turns out to be well in obtaining analytical solution of the fractional order ordi-nary differential equations rapidly. The implementations of difference method such as in the Differential transform method, Abdomian decomposition method (Adomian, 1990) (Adomian and Rach, 1990) (Deng and Wu, 2011) the variational iteration method (Abdou and Soliman, 2005) (Odibat and Momani, 2006) (He and Wu, 2007) (Sweilam, 2007) (Kilbas, *et al.*, 1993) empower us to obtain approximate solution of various ordinary differential equations. Sumudu Transform method (Miller and Ross, 1993) (Podlubny, 1999) (Hilfer, 2000) (Kilbas, *et al.*, 2006) which is newly sub-mitted to the literature is a good technique for solving various kinds of differential equations of arbitrary order. Motivated by the above mentioned work we apply Natural transform method (NTM) on ordinary differential equations of arbitrary order, as this method plays rule of checker on both Laplace and Samudu transform method and has the property to con-verge both Laplace and Samudu transform just by changing variables. We obtain analytical solution of general ordinary differential equations of fractional order. It is illustrated that this article approach that is used to obtain analytical solution of homogenous and nonho-mogeneous problem which will be particularly helpful tool for engineer ,scientist and applied mathematicians.

2. MATERIALS AND METHODS

In this section we provide basic definitions and results from fractional calculus and fundamental properties of Natural transform. For further detailed study, we refer to the reader (Khan and Khan. 2008) (Miller and Ross, 1993) (Podlubny, 1999)

Definition 2.1. The fractional integral of order p $\in R_+$ of a function u $\in L^1([a, b], R)$ is defined as:

$$I_{a+}^{q}u(t) = \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1}u(s)ds.$$

Definition 2.2. The Caputo fractional order derivative of a function u on the interval[a, b] is defined by

$${}^{c}D_{+a}^{p}u(t) = \frac{1}{\Gamma(n-p)}\int_a^t (t-s)^{n-p-1}u^{(n)}(s)ds,$$

where n = [p] + 1 and [p] represents the integer part of p.

Definition 2.3. The Rieman-Liouville fractional order

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derivative of a function on the interval [a, b] is defined

$$D_{a+}^{p}u(t)=\begin{cases}\frac{1}{\Gamma(n-p)}\int_a^t(t-s)^{n-p-1}u(s)ds, & n-1<p<n,\\ \frac{d^n}{dt^n}u(t), & p=n,\end{cases}$$

where n = [p] + 1 and [p] represents the integer part of Definition 2.4. Let g(t) be a function defined for all t ≥ 0.Then Natural Transform of g(t) is the function R(s, u) defined by

$$R(s,u)=N[g(t)]=\int_0^\infty g(ut)e^{-st}dt, \qquad (1)$$

provided that integral on the right exist. Also the inverse N-transform is defined by $g(t)=N^{-1}R(s,u)$.

Theorem 2.5. If g(t) is piece wise continuous in every finite interval $0 \le t \le M$ and of exponential order c for t > M ,then its Natural Transform R(s, u) exist for all s > c, u > c.

Theorem 2.6. [2],If R(s, u) is the Natural Transform of g(t),then the Natural Transform of $g^{(n)}(t)$ is given by

$$N\left[g^{(n)}(t)\right]=\frac{s^n}{u^n}R(s,u)-\sum_{i=0}^{n-1}\frac{s^{n-i-1}}{u^{n-i}}g^{(i)}(t)|_{t=0}$$

where n is integer.

Proof. Let n = 1, then

$$N\left[g^{(1)}(t)\right]=\int_0^a\frac{dg(ut)}{dt}e^{-st}dt$$

$$=\lim_{a\to\infty}\int_0^a\frac{d}{dt}g(ut)e^{-st}dt$$

$$=\lim_{a\to\infty}\left\{\frac{e^{-st}g(ut)}{u}|_0^a+\frac{s}{u}\int_o^a g(ut)e^{-st}dt\right\}$$

$$=\frac{s}{u}R(s,u)-\frac{g(0)}{u}.$$

When n=2, then

$$N\left[\frac{d^2g(t)}{dt^2}\right]=\int_0^\infty g''(ut)e^{-st}dt$$

$$=\lim_{a\to\infty}\int_0^a g''(ut)e^{-st}dt$$

$$=\lim_{a\to\infty}[\frac{g'(ut)e^{-st}}{u}|_o^a+\frac{s}{u}\int_0^a g'(ut)e^{-st}dt]$$

$$=\frac{s}{u}\left[\frac{s}{u}R(s,u)-\frac{g(0)}{u}\right]-\frac{g'(0)}{u}$$

$$=\frac{s^2}{u^2}R(s,u)-\frac{s}{u^2}g(0)-\frac{g'(0)}{u},$$

going on the same process as above,we obtain the Natural Transform of the nth-order deriva-tive as follow

$$N[g^n(t)]=\frac{s^n}{u^n}R(s,u)-\sum_{i=0}^{n-1}\frac{s^{n-i-1}}{u^{n-1}}g^{(i)}(t)|_t=0 \quad (2)$$

$$=s^\beta R(s,u)-\sum_{i=0}^{n-1}s^{i-1}\left[D^{\beta-i}f(t)\right]|_{t=0} \qquad (3)$$

Now we replace s by $\frac{s}{u}$ in(4),we get the Natural Transform of fractional order derivative of g(t) as

$$N\left[D^\beta g(t)\right]=\frac{s^\beta}{u^\beta}R(s,u)-\sum_{i=0}^{n-1}\frac{s^{n-i-1}}{u^{n-i}}D^i[g(t)]|_{t=0}, \quad n-1\le\beta<n. \qquad (4)$$

3. RESULTS AND DISCUSSION

In this section, we discuss the existence of analytical solutions to a general problem of fractional order ordinary differential equation with given initial conditions and we will apply Natural Transform to check the pliability of Natural Transform as follow:

$$D^\beta U(t)+D^2U(t)+DU(t)+U(t)=d \qquad (5)$$

subject to the initial condition $U(0)=g(0)$, $U'(0)=g'(0)$.

Now we will apply the Natural Transform Method (NTM) to the above general problem of ordinary fractional order differential equations with the given initial conditions to obtain analytical solution as:

$$N\left[D^\beta U(t)\right]+N[D^2U(t)]+N[DU(t)]+N[U(t)]=N[d]$$

$$\frac{s^\beta}{u^\beta}R(s,u)-\sum_{i=0}^{n-1}\frac{s^{n-i-1}}{u^{n-i}}[D^ig(t)]_{t=0}+\frac{s^2}{u^2}R(s,u)-\frac{s}{u^2}U(0)$$
$$-\frac{1}{u}U'(0)+\frac{s}{u}R(s,u)-\frac{1}{u}U(0)+R(s,u)=\frac{d}{s} \qquad (6)$$

$$R(s,u)=\frac{\sum_{i=0}^{n-1}\frac{s^{n-i-1}}{u^{n-i}}g^{(i)}(0)+\frac{s}{u^2}g(0)+\frac{1}{u}g(0)+\frac{1}{u}g'(0)+\frac{d}{s}}{\frac{s^\beta}{u^\beta}+\frac{s^2}{u^2}+\frac{s}{u}+1}$$

$$R(s,u)=\frac{h(s,u)+\left(\frac{s}{u^2}+\frac{1}{u}\right)g(0)+\frac{d}{s}}{\frac{s^\beta}{u^\beta}+\frac{s^2}{u^2}+\frac{s}{u}+1}$$

where

$$h(s,u)=\sum_{i=o}^{n-1}\frac{s^{n-1-i}}{u^{n-i}}g^{(i)}(0)+\frac{1}{u}g'(0), \quad (7)$$

taking the inverse Natural Transform of (7) as

$$U(t)=N^{-1}[\frac{h(s,u)+\left(\frac{s}{u^2}+\frac{1}{u}\right)g(0)+\frac{d}{s}}{\frac{s^\beta}{u^\beta}+\frac{s^2}{u^2}+\frac{s}{u}+1}].$$

4 Applications of Natural Transform method to non-homogeneous Fractional ordinary differential Equations

In this portion of the article we have applied Natural to the non-homogeneous fractional ordinary differential equations to derive its analytical solution as bellow.

Example 4.1. We consider the following non-homogeneous fractional ordinary differential equation given by

$$D^\beta U(t)+U(t)=\frac{3}{\Gamma(4-\beta)}t^{3-\beta}-\frac{2}{\Gamma(3-\beta)}t^{2-\beta}+\frac{1}{\Gamma(2-\beta)}t^{1-\beta}+t^3+t^2-t, t>0, 0<\beta\le 2, \quad (8)$$

with the initial conditions being $U(0)=0, U'(0)=0$

In order to solve (8) by using Natural Transform method(NTM),we apply Natural transform on both sides as

$$N[D^\beta U(t)]+N[U(t)]=N\left[\frac{3}{\Gamma(4-\beta)}t^{3-\beta}-\frac{2}{\Gamma(3-\beta)}t^{2-\beta}+\frac{1}{\Gamma(2-\beta)}t^{1-\beta}+t^3+t^2-t\right]$$

$$\frac{s^\beta}{u^\beta}R(s,u)-\sum_{i=0}^{n-1}\frac{s^{n-i-1}}{u^{n-i}}U^i(0)+R(s,u)$$

$$=\frac{3}{\Gamma(4-\beta)}\Gamma(4-\beta)\frac{u^{3-\beta}}{s^{4-\beta}}-\frac{2}{\Gamma(3-\beta)}\Gamma(3-\beta)\frac{u^{2-\beta}}{s^{3-\beta}}+\frac{1}{\Gamma(2-\beta)}\Gamma(2-\beta)\frac{u^{1-\beta}}{s^{2-\beta}}+3\frac{u^3}{s^4}-2\frac{u^2}{s^3}+\frac{u}{s^2}$$

$$\left(\frac{s^\beta+u^\beta}{u^\beta}\right)R(s,u)=\frac{3u^3.s^\beta}{s^4.u^\beta}-\frac{2u^2.s^\beta}{s^3.u^\beta}+\frac{u.s^\beta}{s^2.s^\beta}+\frac{3u^3}{s^4}-\frac{2u^2}{s^3}+\frac{u}{s^2}$$

$$\left(\frac{s^\beta+u^\beta}{u^\beta}\right)R(s,u)=\frac{3u^3}{s^4}\left(\frac{s^\beta+u^\beta}{u^\beta}\right)-\frac{2u^2}{s^3}\left(\frac{s^\beta+u^\beta}{u^\beta}\right)+\frac{u}{s^2}\left(\frac{s^\beta+u^\beta}{u^\beta}\right)$$

$$R(s,u)=\frac{3u^3}{s^4}-\frac{2u^2}{s^3}+\frac{u}{s^2}.$$

Taking the inverse Natural Transform of both sides as

$$N^{-1}[R(s,u)]=N^{-1}\left[\frac{3u^3}{s^4}\right]-N^{-1}\left[\frac{2u^2}{s^3}\right]-N^{-1}\left[\frac{u}{s^2}\right] \quad (9)$$

$$U(t)=t^3-t^2+t.$$

Example 4.3. Consider the following nonhomogeneous ordinary fractional differential equation problem

$$D^{\frac{3}{2}}U(t)=\frac{\Gamma^5}{\Gamma^{\frac{7}{2}}}t^{\frac{5}{2}}-\frac{\Gamma^4}{\Gamma^{\frac{5}{2}}}t^{\frac{3}{2}}+\frac{\Gamma^3}{\Gamma^{\frac{3}{2}}}t^{\frac{1}{2}}+t^2-t+1, t>0 \quad (10)$$

with initial conditions as

$$U(0)=0, \qquad U'(0)=0$$

Now applying the Natural Transform on both sides of (10) as

$$N\left[D^{\frac{3}{2}}U(t)\right]=N\left[\frac{\Gamma^5}{\Gamma^{\frac{7}{2}}}t^{\frac{5}{2}}\right]-N\left[\frac{\Gamma4}{\Gamma\frac{5}{2}}t^{\frac{3}{2}}\right]+N\left[\frac{\Gamma3}{\Gamma\frac{3}{2}}t^{\frac{1}{2}}\right]+N[t^2]-N[t]+N[1]$$

$$\frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}}R(s,u)=\frac{\Gamma5\Gamma\left(\frac{7}{2}\right)u^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)s^{\frac{7}{2}}}-\frac{\Gamma4\Gamma\left(\frac{5}{2}\right)u^{\frac{5}{2}}}{\Gamma\left(\frac{5}{2}\right)s^{\frac{5}{2}}}+\frac{\Gamma3\Gamma\left(\frac{3}{2}\right)u^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)s^{\frac{3}{2}}}+\frac{2u^2}{s^3}-\frac{u}{s^2}+\frac{1}{s}$$

$$R(s,u)=\frac{4u^4}{s^5}-\frac{3u^3}{s^4}+\frac{2u^2}{s^3}+\frac{2u^{\frac{7}{2}}}{s^{\frac{9}{2}}}-\frac{u^{\frac{5}{2}}}{s^{\frac{7}{2}}}+\frac{u^{\frac{3}{2}}}{s^{\frac{5}{2}}}.$$

Now taking the inverse Natural transform as

$$U(t)=N^{-1}[\frac{4s^4}{s^5}-\frac{3u^3}{s^4}+\frac{2u^2}{s^3}+\frac{2u^{\frac{7}{2}}}{s^{\frac{9}{2}}}-\frac{u^{\frac{5}{2}}}{s^{\frac{7}{2}}}+\frac{u^{\frac{3}{2}}}{s^{\frac{5}{2}}}]$$

$$U(t)=t^4-t^3+t^2+\frac{2t^{\frac{7}{2}}}{\Gamma\frac{9}{2}}-\frac{t^{\frac{5}{2}}}{\Gamma\frac{7}{2}}+\frac{t^{\frac{3}{2}}}{\Gamma\frac{5}{2}}$$

5. CONCLUSION AND FUTURE SCOPE

Recently many approaches have been used to obtain approximate solution of some ordinary differential equations of arbitrary order In this article we have solved non-homogeneous ordinary differential equations of arbitrary order by using Natural Transform after describing numerous properties of Natural Transform related to fractional derivative and fractional integral. Natural Transform method can be apply to a variety of initial value problems of fractional order problems in applied mathematics as well as in engineering problems, aeroscience and mathematical biology, biophysics etc. In near future Natural transform method will be applied to many problems of mathematics because of its significant property of being converging to Laplace and Sumudu transforms and plays rule of checker on these two transforms which will enrich the library of fractional calculus.

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