Sindh Univ. Res. Jour. (Sci. Ser.) Vol. 47 (4) 739-742 (2015)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

Proper Homothetic Vector Fields on Three Dimensional Lorentzian Manifold of Circularly Symmetric Static Spacetimes

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Received 12th June 2014 and Revised 11th August 2015

Abstract: In this paper we are interested to investigate proper homothetic vector fields on a three dimensional Lorentzian manifold. For the purpose, Lie derivative of three dimensional circular symmetric static metric has been taken along the vector field. The obtained six coupled partial differential equations are then solved by using direct integration technique. It is investigated that the above metric does not admit proper homothetic vector field .The homothetic vector fields in the above metric are the Killing vector fields.

Keywords: Lorentzian manifold, proper homothetic vector fields, Killing vector fields.

1. INTRODUCTION

In general relativity theory much interest has been shown to study symmetries of spacetime manifolds. These symmetry restrictions are important in finding meaningful physical solutions of the Einstein's field equations. One of the restrictions on a spacetime manifold is interpreted as Killing symmetry. Employing Killing symmetry to a spacetime manifold generates Killing vector fields which in turn give rise to conservation laws (Petrov 1969). Sometimes other symmetries like homothetic or conformal vector fields are imposed on the spacetime metric to study the conservation laws and to get other physical and geometrical features of the spacetime. In four-dimensional spacetime Killing, homothetic and conformal vector fields are investigated by many authors for spherical, plane and axial symmetries. In (Bokhari, *et. al*, 1987) Killing vector fields are obtained for spherically symmetric static spacetime. (Feroze, *et. al*, 2001) classified plane symmetric spacetime according to its Killing vector fields. For spacetime admitting axial symmetry Killing vector fields are investigated in (Misner, *et. al*, 1973) where Kerr-Newman metric is obtained as a solution of Einstein's field equations and two Killing vectors are obtained for this spacetime. (Shabbir and Ramzan, 2007) (Shabbir and Amur, 2006) classified cylindrically symmetric static and Bianchi type-I spacetimes according to their homothetic vector fields. In (Ahmad and Ziad, 1997) the authors obtained homothetic motions of spherically symmetric spacetime. Conformal symmetry for spacetime manifolds are discussed by different authors in (Hall and Steele, 1991) (Kramer and Carot, 1991) (Maartens, *et. al*, 1995).

In literature less work is available on symmetries in three-dimensional metrics. (Bokhari *et. al*, 2008) investigated Killing vector fields on three-dimensional circular symmetric static metric. In (Hall and Capocci, 1999) investigated conformal symmetry for three-dimensional manifolds and the nature of the zeros of the associated vector fields. An interest in three-dimensional gravity is developed because of the study of Black Hole thermodynamics by considering three-dimensional metric (Berredo and Peixoto, 2003) and the Black Hole solutions in three-dimensions by (Banados, *et. al*, 1992). This suggests the important role of three-dimensional spacetime in the explanation of some physical aspects. This fact indeed developed our interest to study proper homothetic vector fields in three-dimensional static circularly symmetric metric. In the following we will classify the above metric to obtain the possible physically significant metrics. A vector field κ is said to be homothetic vector field if it satisfies the relation.

$$\underset{\kappa}{L} g_{\mu\nu} = 2\alpha\, g_{\mu\nu},\ \alpha \in R \setminus \{0\}, \qquad (1)$$

where $\underset{K}{L}$ denotes the Lie derivative with respect to the vector field κ.

2. MAIN RESULTS

Circularly symmetric static metric in its general form with usual coordinates (labeled as $(x^0, x^1, x^2) = (t, r, \theta)$ can be written as (Bokhari *et. al*, 2008)

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2 d\theta^2 \qquad (2)$$

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where A and B are functions of r only. The above metric admits minimum two circular symmetries (Bokhari *et. al*, 2008) as $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \theta}$. In the following we will explore all the possible cases when the above metric admit a proper homothetic vector field. A vector field κ is said to be a homothetic vector field if it satisfies equation (1). Equation (1) can be written explicitly using (2) as

$$A_r(r)\kappa^1 + 2\kappa^0{}_{,0} = 2\alpha, \tag{3}$$

$$e^{B(r)}\kappa^1{}_{,0} - e^{A(r)}\kappa^0{}_{,1} = 0, \tag{4}$$

$$r^2\kappa^2{}_{,0} - e^{A(r)}\kappa^0{}_{,2} = 0, \tag{5}$$

$$B_r(r)\kappa^1 + 2\kappa^1{}_{,1} = 2\alpha, \tag{6}$$

$$e^{B(r)}\kappa^1{}_{,2} + r^2\kappa^2{}_{,1} = 0, \tag{7}$$

$$\frac{1}{r}\kappa^1 + \kappa^2{}_{,2} = \alpha, \tag{8}$$

Differentiating equation (6) with respect to θ we get $2\kappa^1{}_{,12} = -B_r(r)\kappa^1{}_{,2}$. In order to solve this equation we consider two possible cases (λ) $\kappa^1{}_{,2} \neq 0$ and (μ) $\kappa^1{}_{,2} = 0$. We discuss each case in turn.

Case (λ):

In this case $\kappa^1{}_{,2} \neq 0$ and the above equation can be written as $\frac{\kappa^1{}_{,12}}{\kappa^1{}_{,2}} = -\frac{B_r(r)}{2}$. Integrating this equation with respect to r and solving we get

$$\kappa^1 = e^{\frac{-B(r)}{2}}\int F^1(t,\theta)d\theta + F^2(t,r),$$

where $F^1(t,\theta)$ and $F^2(t,r)$ are functions of integration. Using the above equation in (3) and (8) and then solving we get the following system of equations

$$\left.\begin{aligned}
\kappa^0 &= \alpha t - \frac{A_r(r)}{2}\Big(e^{\frac{-B(r)}{2}}\int F^3(\mathrm{t},\theta)\,dt + \int F^2(t,r)\,dt \\
&\quad + F^4(r,\theta),\\
\kappa^1 &= e^{\frac{-B(r)}{2}}F^3(t,\theta) + F^2(t,r),\\
\kappa^2 &= \alpha\theta - \frac{1}{r}\Big(e^{\frac{-B(r)}{2}}\int F^3(\mathrm{t},\theta)d\theta + \theta F^2(\mathrm{t},r)\Big)\\
&\quad + F^5(\mathrm{t},r).
\end{aligned}\right\}$$

where $F^3(\mathrm{t},\theta), F^4(r,\theta)$ and $F^5(\mathrm{t},r)$ are integration functions. In order to get a complete solution of equations (3)-(8) we need to determine the unknown functions , $F^2(t,r)$, $F^3(t,\theta)$, $F^4(r,\theta)$ and $F^5(\mathrm{t},\mathrm{r})$ by using equations (4), (5) and (8). To avoid lengthy and tedious details we will write the resulting vector fields directly as follows:

$$\left.\begin{aligned}
\kappa^0 &= \alpha t - \frac{A_r}{2}\Big(\frac{t^2}{2}F^9(\mathrm{r}) + tF^{10}(\mathrm{r})\Big)\\
&\quad + F^8(\mathrm{r}) + \frac{\theta^2}{2}c_2 + \theta c_3,\\
\kappa^1 &= tF^9(\mathrm{r}) + F^{10}(\mathrm{r}),\\
\kappa^2 &= \alpha\theta - \frac{1}{r}\Big(\mathrm{t}\,\theta F^9(\mathrm{r}) + \theta F^{10}(\mathrm{r})\Big)\\
&\quad + F^5(\mathrm{t},\mathrm{r}),\ \text{where}\, c_2, c_3 \in \Re.
\end{aligned}\right\} \tag{9}$$

Now using equation (7) and doing some calculation we come to know that $\alpha = 0$ and the solution represents Killing symmetry. Thus in case (λ) no proper homothetic vector field exists and the metric takes the form

$$ds^2 = -r^2dt^2 + \frac{c_1}{r^2}dr^2 + r^2d\theta^2. \tag{10}$$

Killing vector fields are

$$\left.\begin{aligned}
\kappa^0 &= c_4,\\
\kappa^1 &= 0,\\
\kappa^2 &= c_5,
\end{aligned}\right\} \tag{10 a}$$

Case (μ):

In this case $\kappa^1{}_{,2} = 0 \Rightarrow \kappa^1 = F^{11}(t,r)$, where $F^{11}(t,r)$ is the function of integration. Using this information in equations (3) and (7) we get the following system of equations:

$$\left.\begin{aligned}
\kappa^0 &= \alpha t - \frac{A_r}{2}\int F^{11}(\mathrm{t},\mathrm{r})\,\mathrm{dt}\\
&\quad + F^{13}(\mathrm{r},\theta),\\
\kappa^1 &= F^{11}(\mathrm{t},\mathrm{r}),\\
\kappa^2 &= F^{12}(\mathrm{t},\theta)
\end{aligned}\right\} \tag{11}$$

where $F^{11}(t,r), F^{12}(t,\theta)$ and $F^{13}(r,\theta)$ are functions of integration which are to be determined. Substituting back the system of equations (11) in equations (5) and (8) respectively, after doing some lengthy calculation, we reach to the following system with three possibilities:

$$\left.\begin{aligned}\kappa^0 &= \alpha t - \frac{A_r}{2}(c_6 A_r^2 t^2 + t c_5 + c_7)\\ \kappa^1 &= c_6 A_r^2 t + c_5,\\ \kappa^2 &= c_8,\end{aligned}\right\} \qquad (11a)$$

$A_r^2(r) \neq 0,\ c_6 = 0,$

(a) $A_r^2(r) = 0,\ c_6 \neq 0,$

(b) $A_r^2(r) = 0,\ c_6 = 0,$

We will discuss each case in turn. Here, we are going to write the results only and details are omitted.

Case $(\mu)(a)$**:**

In this case $A_r^2(r) \neq 0,\ c_6 = 0,$

Using equation (4), one can have $A(r) = rc_{12} + c_{13},\ c_{12}, c_{13} \in \Re(c_{12} \neq 0)$ In this case also no proper homothetic vector fields exist and homothetic vector fields are the Killing vector fields given as

$$\left.\begin{aligned}\kappa^0 &= c_{14}\\ \kappa^1 &= 0,\\ \kappa^2 &= c_8,\end{aligned}\right\} \qquad (12)$$

where $c_8, c_{14} \in \Re$ The metric takes the form

$$ds^2 = -e^{(rc_{12}+c_{13})}dt^2 + e^{B(r)}dr^2 + r^2 d\theta^2, \qquad (13)$$

Case $(\mu)(b)$**:**

In this case $A_r^2(r) = 0,\ c_6 \neq 0,$ In this case metric again does not admit proper homothetic vector field.

Case $(\mu)(c)$**:**

In this case $A_r^2(r) = 0,\ c_6 = 0,$ In this case we get contradiction, so circularly symmetric static metric does not admit proper homothetic vector field.

3. SUMMARY AND DISCUSSION

The general circularly symmetric static metric is classified according to its proper homothetic motions. To obtain a consistent solution we derived six non linear coupled differential equations from equations (1) and (2). In the solution of these six equations there arises two main cases namely (λ) and (μ). Further simplification of these two cases show that the metric in equation (2) does not contain a proper homothetic motion ,but homothetic vector fields are Killing vector fields as shown in equation(12) in case (λ). In the other case (μ) there arise further three cases. Proper homothetic vector fields for the metric (2) are investigated in each of the three cases and it comes out that case $\mu(a)$ admit no proper homothetic motion and Killing symmetries are obtained which are given in equation(12), while cases $\mu(b)$ and $\mu(c)$ again do not give any information about proper homothetic motion.

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