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Investigative Solutions of Viscoelastic Fluid Flow in Channels through Porous Media: Analysis of Velocity

G. Q, MEMON[++], G. H. TALPUR*, P. SHAIKH**, A. W. SHAIKH***

Department of Mathematics, Shah Abdul Latif University, Khairpur Mirs, Sindh, Pakistan.

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Abstract: This paper concerned with solutions of Viscoelastic Flow through Porous Channels for velocity. The associated problem is prepared by means of Darcy-Brinkman model coupled with Oldroyd–B Constitutive Model.The analytical solution is established under the symmetries for the system through Lie group technique. Numerical predictions are found adopting solver ND–Solve function in Mathematica.

Key words: Viscoelastic Flow in channels through Porous Media, Darcy-Brinkman model, Oldroyd–B constitutive model, Lie Group Analysis, Exact and Numerical Solutions concerned with analysis of velocity.

1 INTRODUCTION

In the field of fluid dynamics, investigation of non-linear flows through porous media and mathematical modeling is very difficult work for solving the various challenging problems in applied sciences. The solutions of differential equations governing the compressible or incompressible fluids, Newtonian fluids or non-Newtonian fluids are sensitively involved considerable to attention in the literature. Flows of Newtonian and non-Newtonian fluids related with some essential investigation are prepared by way of Zienkiewicz *et al.* (1966), (Rajagopal and Gupta 1984), (Rajagopal 1985), (Bird *et al.* 1987), (Abel-Malek *et al.* 2002), (Wafo-Soh 2005), (Chen *et al.* 2006), Fetecau and Fetecau (2005, 2007),(Taha and Sochi. 2009, 2010) and others.

The governing equations to model the flow of viscoelastic fluids through porous media, recently developed by Tan and Masuoka (2005), adapts to Darcy–Brinkman model.The objective of this paper is to provide analytical and numerical solution for fluid flow in channels through porous media adopting the constant viscosity Oldroyd–B constitutive model. The Oldroyd–B model describes the viscoelastic behaviour of the stresses under general flow conditions and it is one of the simplest differential constitutive models. See Arada N. Sequeira A (2003).

The symmetries for solving the differential equations are imperative and greater identified than the equations since basic physical laws have been reflected in these symmetries. Symmetries of a differential equations form a one–parameter group of transformations in which parameter is small, and guide to the reduction of the number of independent variables, has been initiated and developed by Olver, (1986), Ibragimov, (19990, Basov, (2004) and others.

The analytical solution is established under the symmetries for the system through Lie group technique. The governing system of partial differential equations (PDEs) is reduced in to a system of ordinary differential equations (ODEs) applying appropriate boundary conditions corresponding to the symmetry group, so that exact invariant solutions can be obtained. Numerical predictions of the system are achieved adopting solver ND–Solve function in Mathematica.

Section 2 deals with the formulation of the problem. Section 3 related with exact solutions of viscoelastic flow in channel filled with porous media, section 3.1 linked with Study State Solution for non-homogeneous equation, section 3.2 includes symmetry analysis for the partial differentials equations 9*(i)* and 9(*iii*). Lie-point symmetries are presented in sections 3.3, section 3.4 connected with results concerned with invariant solution related to the operator X_1 - $\beta X_{3,.}$ Section 4 associated with analysis of velocity, section 4.1 related with analytical solution of velocity, section 4.2 concerned with steady state solution of velocity, section 4.3 contains numerical solution of velocity and section 5 associated with conclusion of the problem.

2 PROBLEM FORMULATIONS

Consider the incompressible laminar flow of viscoelastic fluid in a channel filled with porous medium. The system of governing equations of flow comprises of the conservation of mass and conservation of momentum transport coupled with the Oldroyd–B constitutive model. The flow of viscoelastic fluids

[++]Corresponding Author Email G. Q, MEMON ghutamqadirl0@yahoo.com

*Department of statistics, University of Sindh, Jamshoro.

** Department of Civil Engineering, MUET, Jamshoro

*** IMCS, University of Sindh, Jamshoro.

through porous media is assumed to be isotropic and homogeneous. The momentum equation can be modelled by using Darcy-Brinkman model and in the absence of body force, equations of continuity and momentum may be written in the following form:

$$\nabla.\bar{u} = 0 \qquad (1)$$

$$\frac{\rho}{\varepsilon}\frac{\partial \bar{u}}{\partial t} = \frac{1}{\varepsilon}\nabla([2\mu_2 \underline{\underline{d}}] + \tau) - \nabla p - \rho\bar{u}.\nabla\bar{u} - \frac{\mu}{K}\bar{u} \qquad (2)$$

The Oldroyd–B constitutive equation describes the viscoelastic stresses in the flow can be expressed as below:

$$\lambda\frac{\partial \tau}{\partial t} = [2\mu_1 \underline{\underline{d}}] - \tau - \lambda\{\bar{u}.\nabla\tau - \nabla\bar{u}.\tau - (\nabla\bar{u})^T.\tau\} \qquad (3)$$

In the above equations, u is the velocity vector field of flow, τ is the extra stress tensor, $\underline{\underline{d}}$ is the rate of strain tensor, ∇ is the spatial differential operator, p is the isotropic fluid pressure (per unit density) and t is the time. The μ_1 and μ_2 are respectively the viscoelastic solute and Newtonian solvent viscosities, fluid density is denoted by ρ, whereas λ is the relaxation time of the viscoelastic fluid and K is the intrinsic permeability of the porous medium. Total viscosity μ of the viscoelastic flow is $\mu = \mu_1 + \mu_2$ and is taken constant. In the above equation (2), the acceleration co–efficient tensor is assumed to be $1/\varepsilon$ and ε is porosity of porous media. The equations are derived which govern the unsteady unidirectional flow of viscoelastic fluid through porous media adopting Oldroyd–B constitutive model. For unidirectional flow the velocity field is $\bar{u} = (0, v(t,y), 0)$; here the description of velocity automatically gives pleasure to the incompressibility state. The derivation of such equations by employing the momentum transport equation of viscoelastic fluid and Oldroyd–B constitutive equations assuming constant pressure gradient and may be expressed in the absence of body force and the governing system of equations is written in the dimensionless form as:

$$\left.\begin{aligned} &\mathrm{Re}\frac{\partial v}{\partial t} = 1 + \mu_2\frac{\partial^2 v}{\partial y^2} + \frac{\partial \tau_{12}}{\partial y} - \frac{1}{Da}v && (i),\\ &We\frac{\partial \tau_{11}}{\partial t} = 2We\,\tau_{12}\frac{\partial v}{\partial y} - \tau_{11} && (ii)\\ &We\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial v}{\partial y} - \tau_{12} && (iii) \end{aligned}\right\} \quad (4)$$

Where $v(y,t)$, $\tau(y,t)$ and y are dimensionless velocity, stress tensor and transversal coordinates and t is the non-dimensional time and the dimensionless Reynolds number (Re), Weissenberg number (We) and Darcy's number (Da) are Re $= \rho\, L\, Vc\, /\, \mu$, We $= \lambda^* = \lambda\, Vc\, /$L and $Da = K^*/\varepsilon\, L^2$ respectively. in which K is the non-dimensional modified permeability of the porous medium. Whilst, L is the characteristic length taken as half width of the channel and Vc is the characteristic velocity assumed as reference axial velocity $Vc = \varepsilon\, L^2\, (-\partial p/\partial x)\, /\mu$,

To complete the well posed problem specification, it is necessary to prescribe initial and boundary conditions. So initial conditions are taken from rest as:

$v(0, y) = 0$, $\tau_{11}(0, y) = 0$,. $\tau_{12}(0, y) = 0$,. when $t > 0$ (5)

and boundary conditions are taken as:

$v(t, -1) = 0$ and $v(t, 1) = 0$, when $-1 \leq y \leq 1$ (6)

3 Exact solutions of viscoelastic flow in channels filled with porous media

3.1 Steady State Solution for non-homogeneous equation

For finding the study state solution of non–homogeneous equation (4(i)), we consider

$$\left.\begin{aligned} &v(t,y) = u_1(t,y) + \phi(y), \quad \tau_{11}(t,y) = u_2(t,y) + \psi(y)\\ &\& \quad \tau_{12}(t,y) = u_3(t,y) + \varphi(y) \end{aligned}\right\} 7$$

Substituting (7) into PDE's system (4) reduces ODE's system of $\varphi(y), \psi(y)$ *and* $\phi(y)$ and applying the boundary conditions $\varphi(-1) = 0$ and $\varphi(1) = 0$ (10), the steady-state solutions is given as under

$$\left.\phi(y) = Da\,(1 - \frac{Cosh\frac{y}{\sqrt{Da}}}{Cosh\frac{1}{\sqrt{Da}}}),\quad \psi(y) = 2We\,\mu_1\,Da\,\frac{Sinh^2\frac{y}{\sqrt{Da}}}{Cosh^2\frac{1}{\sqrt{Da}}},\quad \varphi(y) = -\mu_1\sqrt{Da}\,\frac{Sinh\frac{y}{\sqrt{Da}}}{Cosh\frac{1}{\sqrt{Da}}}\right\} \qquad (8)$$

and after separating, second system which is PDE's is given as:

$$\left.\begin{array}{ll} Re\,\dfrac{\partial u_1}{\partial t}=\mu_2\dfrac{\partial^2 u_1}{\partial y^2}+\dfrac{\partial u_3}{\partial y}-\dfrac{1}{Da}u_1 & (i),\\ We\,\dfrac{\partial u_2}{\partial t}+u_2=2We\,\{u_3+\phi(y)\}\dfrac{\partial u_1}{\partial y}+2We\,\varphi'(y)\,u_3 \quad (ii), & We\,\dfrac{\partial u_3}{\partial t}+u_3=\mu_1\dfrac{\partial u_1}{\partial y} \quad (iii) \end{array}\right\} \tag{9}$$

Subject to boundary and initial conditions is written as:

$u_1(t,-1)=0,$ (10- i) and $u_1(t,1)=0,.$ (10-ii) when t > 0

$$u_1(0,y)=-\,Da\,\left(1-\frac{Cosh\frac{y}{\sqrt{Da}}}{Cosh\frac{1}{\sqrt{Da}}}\right),. \quad (10\text{-iii}) \qquad u_2(0,y)=-2We\,\mu_1\,Da\,\frac{Sinh^2\frac{y}{\sqrt{Da}}}{Cosh^2\frac{1}{\sqrt{Da}}}, \tag{10}$$

$$u_3(0,y)=\mu_1\,\sqrt{Da}\,\frac{Sinh\frac{y}{\sqrt{Da}}}{Cosh\frac{1}{\sqrt{Da}}} \quad (10\text{-v}) \quad \text{when } -1\le y\le 1 \text{ and } \mu_1+\mu_2=1$$

3.2 Symmetry analysis for the partial differentials equations

Once Lie group symmetry of the differential equation is known, it can be used in the investigation of transformations that will reduce the equation to simpler form and it is powerful method in obtaining analytical solutions of differential equations. In this section, symmetry conditions and method for finding the Lie point symmetries of the equations (9(*i*)) and (9(*iii*)) (because derivatives of these equations are connected each other) are introduced. The operator

$$X=\varphi(t,y,u_1,u_3)\frac{\partial}{\partial t}+\xi(t,y,u_1,u_3)\frac{\partial}{\partial y}+\eta^1(t,y,u_1,u_3)\frac{\partial}{\partial u_1}+\eta^2(t,y,u_1,u_3)\frac{\partial}{\partial u_3} \tag{11}$$

is the Lie point symmetry generator for partial differentials equations (9(*i*)) and (9(*iii*)) iff,

$$X^{[2]}\left(\mu_2\,u_{1yy}+u_{3y}-\frac{u_1}{Da}-Re\,u_{1t}\right)\Big|_{(9-i\ \&\ iii\ )}=0 \quad \Rightarrow \frac{-1}{Da}\eta^1-Re\,\eta_t^{1[1]}+\eta_y^{2\,[1]}+\mu_2\eta_{yy}^{1[2]}\Big|_{(9-i\ \&\ iii\ )}=0$$

$$\text{and}\quad X^{[1]}(\mu_1 u_{1y}-We\,u_{3t}-u_3)\Big|_{(9-i\&iii)}=0 \quad \Rightarrow -\eta^2+\mu_1\eta_y^{1[1]}-We\,\eta_t^{2\,[1]}\Big|_{(9-i\&iii)}=0 \tag{12}$$

Where first and second extended infinitesimal generator of X are

$$X^{[1]}=X+\eta_t^{1[1]}\frac{\partial}{\partial u_{1t}}+\eta_y^{1[1]}\frac{\partial}{\partial u_{1y}}+\eta_t^{2[1]}\frac{\partial}{\partial u_{3t}}+\eta_y^{2[1]}\frac{\partial}{\partial u_{3y}}\ , \qquad X^{[2]}=X^{[1]}+\eta_{yy}^{1[2]}\frac{\partial}{\partial u_{1yy}} \tag{13}$$

In which

$$\eta_t^{1[1]}=D_t\eta^1-u_{1t}\,D_t\phi-u_{1y}\,D_t\xi\ ; \qquad \eta_y^{1[1]}=D_y\eta^1-u_{1t}\,D_y\phi-u_{1y}\,D_y\xi\ ;$$
$$\eta_t^{2[1]}=D_t\eta^2-u_{3t}\,D_t\phi-u_{3y}\,D_t\xi\ ; \qquad \eta_y^{2[1]}=D_y\eta^2-u_{3t}\,D_y\phi-u_{3y}\,D_y\xi\ ;$$
$$\eta_{yy}^{1[2]}=D_y\eta_y^{[1]}-u_{1ty}\,D_y\phi-u_{1yy}\,D_y\xi \tag{14}$$

Where Dx_i is the total derivative operator.

3.3 Lie-point symmetries of the PDE's (9(*i*)) and (0(*iii*))

The expansion of (14) could be put in the form of symmetry (12). Subsequent to equating the partial differentials of u_1, u_3 and powers of differentials of these equations, the operators can be found after making simpler the over resolved linear partial differentials equations systems, as over determined systems of linear partial differentials is described and after solving these two over determined systems of linear PDE's, a result of linear equating systems presents values of the functions ϕ, ξ, η^1 and η^2 and re written as under:

$$\phi = c_1,\ \xi = c_2,\ \ \eta^1 = c_3\, u_1 + f\,(t, y) \ \text{ and } \eta^2 = c_3\, u_3 + g\,(t, y) \tag{15}$$

Where $f(t, y)$ & $g\,(t, y)$ are arbitrary functions of the following equations,.

$$\mathrm{Re}\ f_t = \mu_2\ f_{yy} + g_y - \frac{1}{Da} f \ \text{ and } We\ g_t = \mu_1\ f_y - g$$

In (15), c_i are arbitrary constants and symmetry Lie algebra of PDE's system (9(*i*)) and (9(*iii*)) is three-dimensional and spanned by the following generators

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = \frac{\partial}{\partial y},\qquad X_3 = u_1 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_3} \ \text{ and } X_m = f\,(t,\ y) \frac{\partial}{\partial u_1} + g\,(t,\ y) \frac{\partial}{\partial u_3} \tag{16}$$

Where *m* are non-negative numbers

3.4 Results concerned with invariant solutions related to X_1 - βX_3

In this section only those operators are used that related to solution of physical problem (9(*i*)) and (9(*iii*)) subject to same initial and boundary conditions. Method of solutions depends on the applications of Lie group of transformation related with one-parameter to the PDE's system. As $X = X_1 - \beta X_3 = \frac{\partial}{\partial t} - \beta \frac{\partial}{\partial u_1} - \beta \frac{\partial}{\partial u_3}$

Hence applying the operator *X* and applying the super position principle, the final result of the system (4) to (6) show the following solutions

$$\left.\begin{aligned}
v\,(t,\ y) &= \sum_{n=1}^{\infty} \frac{2\,Da\ (-1)^{n-1}}{\lambda_n\,(Da\ \lambda_n^{\,2} + 1)} \left(\frac{\beta_2\,(1-\beta_1 We)}{(\beta_1 - \beta_2)} e^{-\beta_1 t} - \frac{\beta_1\,(1-\beta_2 We)}{(\beta_1 - \beta_2)} e^{-\beta_2 t} \right) Cos\,\lambda_n\, y + Da\ \left(1 - \frac{Cosh\ \frac{y}{\sqrt{Da}}}{Cosh\ \frac{1}{\sqrt{Da}}}\right) \\
\tau_{11}(t,\ y) &= \left(\sum_{n=1}^{\infty} \frac{Da\ (8\,We\ \ \mu_1)^{\frac{1}{2}}}{(Da\ \lambda_n^2 + 1)} \left(\begin{aligned} &\frac{\beta_2^2\,(1-\beta_1 We)\ (e^{-2\beta_1 t} - e^{\frac{-t}{We}})}{(\beta_1 - \beta_2)^2\ (1 - 2\,\beta_1 We)} + \frac{\beta_1^2\,(1-\beta_2 We)\,(e^{-2\beta_2 t} - e^{\frac{-t}{We}})}{(\beta_1 - \beta_2)^2\,(1 - 2\,\beta_2 We)} \\ &- \frac{\beta_1 \beta_2\,(2 - \beta_1 We - \beta_2 We)\ (e^{-(\beta_1 + \beta_2) t} - e^{\frac{-t}{We}})}{(\beta_1 - \beta_2)^2 (1 - \beta_1 We - \beta_2 We)} + \frac{\beta_2\ (2 - \beta_1 We)\ (e^{-\beta_1 t} - e^{\frac{-t}{We}})}{(\beta_1 - \beta_2)(1 - \beta_1 We)} \\ &- \frac{\beta_1\,(2 - \beta_2 We)\ (e^{-\beta_2 t} - e^{\frac{-t}{We}})}{(\beta_1 - \beta_2)(1 - \beta_2 We)} \end{aligned} \right)^{\frac{1}{2}} Sin\ \lambda_n\, y \right)^2 \\
&\quad + 2 We\,\mu_1\,Da\ (1 - e^{\frac{-t}{We}})\ \frac{Sinh^2 \frac{y}{\sqrt{Da}}}{Cosh^2 \frac{1}{\sqrt{Da}}} \\
\tau_{12}(t,\ y) &= = \sum_{n=1}^{\infty} \frac{2\,\mu_1\,Da\ (-1)^n}{(Da\ \lambda_n^{\,2} + 1)} \left(\frac{\beta_2}{(\beta_1 - \beta_2)} e^{-\beta_1 t} + \frac{\beta_1}{(\beta_1 - \beta_2)} e^{-\beta_2 t} \right) Sin\,\lambda_n\, y - \mu_1\ \sqrt{Da}\ \frac{Sinh\ \frac{y}{\sqrt{Da}}}{Cosh\ \frac{1}{\sqrt{Da}}}
\end{aligned}\right\} \tag{17}$$

Where $\lambda_n = (\frac{2n-1}{2})\pi$ and $\mu_1 + \mu_2 = 1$ and

$$\beta_1 = \frac{1}{2}\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right)^2 - \frac{4\,(Da\,\lambda_n^2 + 1)}{Da\,\mathrm{Re}\,We}}$$

$$\beta_2 = \frac{1}{2}\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right) - \frac{1}{2}\sqrt{\left(\frac{1}{We} + \frac{\mu_2\,\lambda_n^2}{\mathrm{Re}} + \frac{1}{Da\,\mathrm{Re}}\right)^2 - \frac{4\,(Da\,\lambda_n^2 + 1)}{Da\,\mathrm{Re}\,We}}$$

4 Analysis of Velocity

4.1 Analytical solution of velocity

As analytical result of velocity has been found by symmetry method and is written in the following form

$$v\,(t\,,\,y)=\sum_{n=1}^{\infty}\frac{2\,Da\;(-1)^{n-1}}{\lambda_n\,(\,Da\;\lambda_n^{\,2}+1)}\left(\;\frac{\beta_2\,(1-\beta_1\,We)}{(\beta_1-\beta_2)}\;e^{-\beta_1\,t}-\frac{\beta_1\,(1-\beta_2\,We)}{(\beta_1-\beta_2)}\;e^{-\beta_2\,t}\right)\,Cos\,\lambda_n\,y+Da\;\left(1-\frac{Cosh\,\frac{y}{\sqrt{Da}}}{Cosh\,\frac{1}{\sqrt{Da}}}\right)\qquad(18)$$

This solution (18) is expressed in figure-1 for several of parameters and at different time *t*.

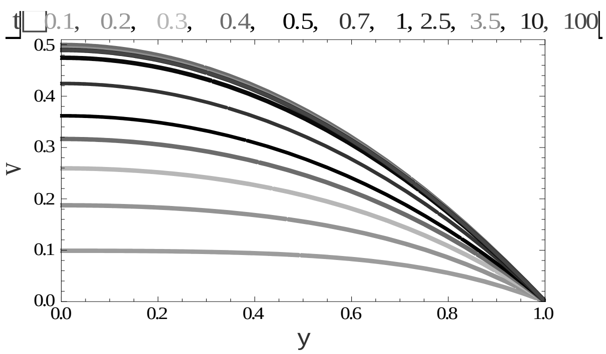

Fig.1: Analytical solution of the velocity *v* of (18) with *Re* = 1, *Da*=20 and at different time *t*

The time dependent effect on the axial velocity is displayed in figure–1 respectively. **(Fig.1)** illustrate that the channel velocity *v* increase as the time increase and there is an upper limit for these increases as velocity reaches at v = 0.5 and then it decreases and reaches at steady–state whose steady state value is v = 0.491and there is no further change in the channel velocity

4.2 Steady State Solution of velocity

Steady state solution of velocity has been already solved and given in the relation (8) *i.e*.

$$v(t,y)=\phi(y)=Da\;\left(1-\frac{Cosh\,\frac{y}{\sqrt{Da}}}{Cosh\,\frac{1}{\sqrt{Da}}}\right)\qquad(19)$$

Steady–state solutions (19) are plotted in figures 2 at different values of Darcy's number *Da*.

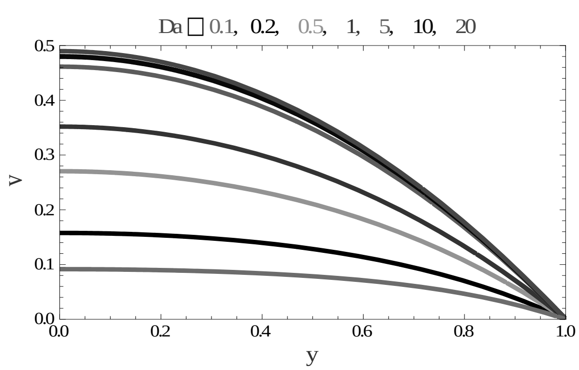

Fig..2: Steady–state solution of the velocity (19) with $\mu = 1$ at different values of *Da*.

The steady–state velocity is displayed in **(Fig.2,3)** respectively. The fig,2 shows that in the steady–state, if channels having small Darcy's numbers Da, then steady velocity *v* have small values that if permeability decreases, then resistance increases and hence velocity decreases in the steady–state.

4.3 Numerical solutions of viscoelastic flow in channels filled with porous media.

Numerical solution are obtained for the system of partial differential equations (4) subject to initial and boundary conditions (5) and (6) by using Mathematica solver NDSolve and solution of velocity is plotted in the **(Fig.3)** with increasing time.

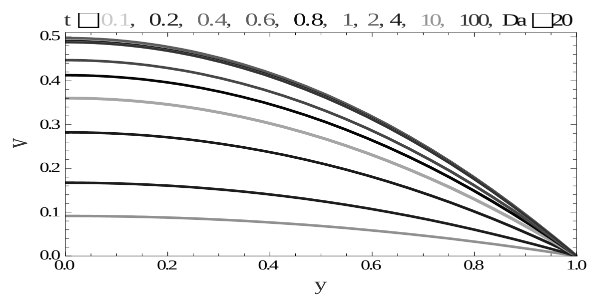

Fig. 3: Numerical solution of the velocity *v* with $Da = 20$, $Re = 1$, , $\mu_1 = 1/9$, $\mu_2 = 8/9$, $We = 1$ **and at different time *t***

The result of numerical solutions of velocity is displayed in figure–3 respectively. Figure–3 shows that as time proceeds, the channel velocity increases and there is an upper limit for this increase whose maximum value is *v* = 0.5 and then it decreases and reaches at steady–state, then its value is *v* = 0.491as time approaches beyond six units (*t* > 5). The numerical results are comparable with analytical solutions.

5 CONCLUSION

The main purpose of our research to solve the problem of PDE's system for the analysis of velocity and Lie group technique has been applied successfully for solving system of PDEs of flow of viscoelastic fluid in channels through porous medium coupled with constant viscosity Oldroyd–B constitutive model. In this paper, firstly steady state solution have been obtained. The analytical solution of PDE's system associated with new dependent variables is founded under the lie-point symmetries through Lie group technique, so that accurate invariant results have been found. This technique can give some practical approaching interested in include of results and may support for determining exacting results in several cases. Also, the analytical solution, numerical solution of the velocity and steady–state velocity is displayed in figures and discussed. The numerical solution is obtained through Mathematica by ND–Solver which is in close agreement with analytical solutions. In both analytical and numerical techniques, comparison of both techniques was conducted. We hope that the results may be useful for other workers in the field. Our work are developing and putting into practice other solution and transient viscoelastic algorithms.

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