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Numerical solutions of Fractional Order System of Bagley-Torvik Equation Using Operational Matrices

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Abstract: In this article, we establish a numerical scheme for the numerical solutions of Bagaly Torvik equations. For the numerical scheme, we construct some operational matrices by using Shifted Legendre polynomials. The considered system is shifted to easily soluble algebraic equations which is easily computable by using Matlab. We give numerical simulations and comparison also which demonstrate accuracy of the method.

Keywords: Fractional differential equations; Numerical solutions, boundary conditions Legendre polynomials, Operational matrices.

1. INTRODUCTION

Differential equations of fractional order provide an excellent tools for the description of many engineering, physical and biological phenomenons. The researchers of various scientific and engineering disciplines such as, mechanics, chemistry, viscoelasticity, control theory, signal and image processing phenomenon, economics, optimiza-tion theory etc are taking interest to study them for detail see articles of Kilbas *et.al* (1993), Miller *et.al* (1993), Podlubny (1999), Hilfer (2000), Kilbas and Srivastava *et.al* (2006) and the references there in. Due to these importance and applications of fractional order differential equations, engineers are taking interest in the studies of frictional differential equations because its models are more realistic and practical. In this paper, we study Bagley Torvik equation which was originally formulated in the studies on behavior of real material by the use of fractional calculus Torvik and Bagley (1983). The applications of Bagley Torvik system. Bagley and Torvik (1983) has been found in many engineering and applied science problems. Specially the equations of half and one-third order derivative have significant importance for engineers. Because it describes motion of real physical systems, the modeling of the motions of the rigid plate immersed in a Newtonian fluid and a gas in a fluid respectively kantham *et. al* (2009).This equations has been studied by many authors. In this regard an ap-proximate analytical solution of the said equation was derived by means of Adomain decomposition method by Ray and Bera (2005), variational iteration method A. Ghorbani and Alavi (2008), Taylor collocation method. Cenesiz *et.al* (2010) and Evolutionary computational intelligence method by Nemati *et.al* (2013). Diethem (2002) transformed the equations into first order coupled fractional differential equation and solved the problem with Adams Bashforth predictor corrector methods. Podlubny (2009) solved the Bagley-Torvik equations by successive approximation method numerically. The same Fractional differential equations was also solved by matrix approach to discretization of fractional derivative by Podlubny *et. al* (2009). But to the best of our knowledge the polynomial spectral methods have not applied to find the numerical solutions of the said problem. In this paper we solve the Bagley-Torvik fractional differential equations by means of shifted Legendre polynomials methods whose generic form is given by

$$A\frac{d^2Y}{dt^2}+B\frac{d^\alpha Y}{dt^\alpha}+C[Y]^n = f(t), 0<\alpha\le 1, \quad (1.1)$$

$$\frac{\mathrm{d}^\mathrm{m}}{dt^m}Y(0)=a_m,\ \frac{\mathrm{d}^\mathrm{m}}{dt^m}Y(T)=b_m, m=0,1, 0\le t\le T.$$

Where $f(t)$ is source term of the systems, n is non-linear operator of the equations, $Y(t)$ is the solutions of equation, A,B,C are constants coefficients, T is the constants representing the span of input within the closed interval $[0,T]$, and ak,bk are the constants of the systems. The general response equations(1) containing the parameters whose various response for varying the values of the parameter. When $n = 1, A = M$, mass of the rigid plate, $C = \kappa$, the stiffness of the spring, $B = 2S\sqrt{\mu\rho}$, where S is area of plate immersed in Newtonian fluid, μ is viscosity, and ρ is the fluid density, then (1) represents the motion of a large thin plate in a Newtonian fluid. Similarly damped fractional oscillator with the damping term has a fractional derivative of $\alpha = 1.5$, and it can be represented by Bagley-Torvik fractional differential equations has gained much

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attentions from researchers. Motivated by the above mentioned work, we develop a numerical scheme based on spectral method derived by means of shifted Legendre polynomials.

2. MATERIALS AND METHODS

In this section we provide some fundamentals definitions and known results of fractional calculus and Functional analysis which are found in Kilbas *et.al* (1993), Pudlubny (1999) and Saadatmandi and Deghani (2010).

Definition 2.1. The fractional integral of order $\alpha \in R_+$ of a function $Y \in L^1([a, b], R)$ is defined as:

$$I_{a+}^{\alpha} Y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} Y(s) ds.$$

Definition 2.2. The Caputo fractional order derivative of a function Y on the interval[a, b] is defined by

$${}^{c}D_{+a}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where n = [α] + 1 and [α] represents the integer part of α.

Lemma 2.1 The fractional differential equation of order $\alpha > 0$

$$D^{\alpha} Y(t) = 0, \; n-1 < \alpha \leq n,$$

has a unique solution of the form $Y(t) = c_0 + c_1 t + c_2 t + \ldots + c_{n-1} t^{n-1}$, where $c_k \in R$ and $k = 0, 1,2,3,\ldots,n-1$.

Lemma 2.2 The following result holds for fractional differential equations

$$I^{\alpha}(D^{\alpha}(t)) = Y(t) + c_0 + c_1 t + c_2 t + \ldots + c_{n-1} t^{n-1},$$

for arbitrary $c_k \in R$, k=0,1,2,...,n−1. Hence it follows that

$$D^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, D^{\alpha}[C] = 0.$$

and $$I^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} t^{k+\alpha}.$$

2.3 The Legendre polynomials. In this subsection we call the well know Legendre polynomials defined on [-1,1], which is given by the following relation (2.1)

$$P_{i+1} = \frac{2i+1}{i+1} x\, P_i(x) - \frac{i}{i+1} P_{i-1(x),} \quad i=1,2,3\ldots,$$

Where $P_0(x) = 1, P_1(x) = x.$

By the use of transformation $t = \frac{x+1}{2}$ the interval [-1,1] is shifted to [0,1] and shifted Legender polynomails defined on [0,1] given by

$$(2.2) \quad P_j = \sum_{k=0}^{J} \frac{(-1)^{j+k}\Gamma(j+k+1)}{\Gamma(j-k+1)\Gamma^2(k+1)} t^k, \; j = 1,2,3,\ldots,$$

Where $P_j(0) = (-1)^j, P_j(1) = 1.$

The orthogonality condition is

$$(2.3) \quad \int_0^1 P_n(t) P_m(t)\, dt = \begin{cases} \frac{1}{2m+1}, & if\ m = n, \\ 0, & if\ m \neq n, \end{cases}$$

which implies that any function $f(t) \in C[0,1]$ can be approximated by shifted Legendre polynomails as

$$(2.4) \quad f(x) \approx \sum_{j=0}^{m} C_j\, P_j(t), \quad \text{where} \quad C_j = (2j+1) \int_0^1 f(t) P_j(t) dt.$$

In vector notation (2.4) can be written as

$$(2.5) \quad f(t) \approx K_M^T \psi_M(t),$$

Where $M = m+1, K_M \in R^{M+1}$ is the coefficient vector and ψ_M is M terms function vector.

Lemma 2.5 Convergence analysis for error; (i) Let $f(t) \in C^{M+1}[0,T]$, then $K_M^T \psi_M(t)$ is the best approximation of $f(t)$ on [0,T], then the error approximation is given by

$$\|\boldsymbol{f}(\boldsymbol{t}) - K_M^T \psi(t)\|^2 \leq \frac{1}{4M^{M+1}} \max_{t\in[0,1]} \left|f^{(M+1)}(t)\right|.$$

(ii) Let $f(t) \in Y$, where Y is spanning set of first $M+1$ Legendre polynomials and

$$f(x) = \sum_{i=0}^{M} C_i\, P_i^T(t), \quad \text{then} \quad |c_i| \approx \frac{C}{(\delta_i)^M} \|f_M\|, \quad \text{and}$$

$$\|f(t) - \sum_{i=0}^{M} C_i P_i^T(t)\|^2 = \sum_{i=M}^{\infty} \varepsilon_i C^2,$$

where $\varepsilon_i = i(i+1)$ and $C_i = \frac{2j+1}{T} \int_0^T y(t) P_i^T(t) dt,$

where C is constant**.**

Proof: Its proof is given in Nemati., Ordokhani (2013).

3. OPERATIONAL MATRICES OF INTEGRATION AND DIFFERENTIATIONS

In this section we provide some operational matrices for fractional order integration and fractional order differentiation as in [14,20,29]. We give the following lemma for further process.

Lemma 3.1. Let $\psi_M(t)$ be the function vector as defined in (2.5) then the integrations of order α of φ is defined by $I^{\alpha}\psi_M(t) \approx G_{M\times M}^{\propto} \psi_M(t)$, where $G_{M\times M}^{\propto}$ is the operational matrix of integrations of order $\propto$ and given by

$$G^{\alpha}{}_{M\times M} = \begin{bmatrix} \sum_{k=0}^{0} \Omega_{0,0,k} & \cdots & \sum_{k=0}^{0} \Omega_{m,0,k} \\ \vdots & \ddots & \vdots \\ \sum_{k=0}^{m} \Omega_{m,0,k} & \cdots & \sum_{k=0}^{m} \Omega_{m,m,k} \end{bmatrix}, \quad (3.1)$$

where $\Omega_{i,j,k} =$

$$(2j+1) \sum_{i=0}^{j} \frac{(-1)^{i+j+k+l}\Gamma(i+k+1)\Gamma(l+j+1)}{\Gamma(i-k+1)\Gamma(k+1)\Gamma(k+\propto+!)\Gamma(j-l+1)\Gamma^2(l+1)(k+l+\propto+1)}$$

Proof: For the proof of this lemma see Khalil and Khan (2014).

Lemma 3.2. Let $\psi_M(t)$ be the function vector as defined in (2.5) then the integrations of order α of $\psi_M(t)$ *w.r.t t* is defined by
$D^{\alpha}\psi_M(t) \approx \psi_M(t), H^{\propto}_{M\times M}$, where
$H^{\alpha}{}_{M\times M}$ is the operational matrix of differentiation of order α and given by

$$H^{\alpha}{}_{M\times M} = \begin{bmatrix} \sum_{k=0}^{1}\Phi_{1,1,k} & \cdots & \sum_{k=0}^{1}\Phi_{1,m,k} \\ \vdots & \ddots & \vdots \\ \sum_{k=0}^{m}\Phi_{m,1,k} & \cdots & \sum_{k=0}^{m}\Phi_{m,m,k} \end{bmatrix}, \quad (3.3)$$

where (3.4)

$$\Phi_{i,j,k} = (2j+1)\sum_{k=\lceil\propto\rceil}^{j} \frac{(-1)^{i+j+k+l}\Gamma(i+k+1)\Gamma(l+j+1)}{\Gamma(i-k+1)\Gamma(k+1)\Gamma(k-\propto+!)\Gamma(j-l+1)\Gamma^2(l+1)(k+l-\propto+1)}$$

$j = 0,1,2,\dots,m$ where $\Phi_{i,j,k} = 0$, if $j < \alpha$.

Proof: The proof of this lemma similar as above lemma.

Lemma 3.3. Let $\varphi(t)$be any function defined in [0,T], and $Y(t) = K_M^T\ \psi_M(t)$ then

$$\varphi(t)\mathrm{I}^{\propto}\mathrm{Y}(t) = \mathrm{K}_{\mathrm{M}}^{\mathrm{T}}\ \mathrm{Q}_{\mathrm{M}\times\mathrm{M}}^{(\propto,\varphi)}\ \psi_M(t), \quad (3.5)$$

where $Q^{(\alpha,\varphi)}$ is the operational matrix and is defined as

$$Q^{(\alpha,\varphi)} = \begin{bmatrix} \Phi_{0,0} & \cdots & \Phi_{0,m} \\ \vdots & \ddots & \vdots \\ \Phi_{0,m} & \cdots & \Phi_{m,m} \end{bmatrix}, \quad (3.6)$$

where

$$\Phi_{i,j} = (2j+1)\Delta_i \int_0^T \varphi(t)P_j(t)dt, \quad (3.7)$$

$$\Delta_i = \sum_{l=0}^{i} \frac{(-1)^{i+l}\Gamma(i+l+1)}{\Gamma(i-l+1)\Gamma(l+1)\Gamma(\propto+l)}. \quad (3.8)$$

Proof: Consider the general term of $\psi_M(t)$,

$$I^{\propto}P_i(t) = \frac{1}{\Gamma(\propto)}\int_0^T (T-s)^{\propto-1}P_i(s)ds, \quad (3.9)$$

using the definition of shifted Legendre polynomials, we get (3.10)

$$I^{\propto}P_i(t) = \sum_{l=0}^{i}\frac{(-1)^{i+l}\Gamma(i+l+1)}{\Gamma(i-l+1)\Gamma^2(l+1)\Gamma(\propto)}\int_0^T (T-s)^{\propto-1}s^t ds.$$

To evaluate the integral, we use the well known Laplace transform, that is

(3.11) $\mathcal{L}(\int_0^T (T-s)^{\propto-1}s^l ds) = \frac{\Gamma(\propto)\Gamma(l+1)}{s^{(\propto+l+1)}}$.

Using the inverse laplace transform we get

(3.12) $\int_0^T (T-s)^{\propto-1}s^t ds = \frac{\Gamma(\propto)\Gamma(l+1)T^{\propto+1}}{\Gamma(\propto+l)}$.

In view of (3.11) and (3.12) we can write (3.10) as (3.13)

$$I^{\propto}P_i(t) = \sum_{l=0}^{i}\frac{(-1)^{i+l}\Gamma(i+l+1)}{\Gamma(i-l+1)\Gamma(l+1)\Gamma(\propto+l)} = \Delta_i.$$

Using (3.13), we can write (3.10) as

(3.14) $\varphi(t)I^{\propto}P_i(t) = \Delta_i\varphi(t)$.

Which can be further be written in term shifted Legendre polynomails as

(3.15) $\Delta_i\varphi(t) = \sum_{j=0}^{m}\varphi_{i,j}P_j(t)$,

where $\varphi_{i,j}$ can be calculated using relation

(3.16) $\varphi_{i,j} = (2j+1)\int_0^T \Delta_i\varphi(t)P_i(t)dt$.

Now evaluating the above relation for $i = 0,1,2,3\dots m$ and $= 0,1,2,3\dots\dots m$, we can get the desired result.

4 RESULTS AND DISCUSSION

In this section we construct the new numerical procedure by mean of shifted Legendre polnomails and operational matrices developed in Section (3).

4.1. Bagley-Torvik equation. Consider the following Bagley-Torvik equations involving the fractional order differentiation

Case I

(4.1) $AD^2Y(t) + BD^{\propto}Y(t) + CY(t) = f(t), \quad 1 < \propto \leq 2, \ 0 < t \leq T$

Subject to the boundary conditions $Y(0) = a_0, \ Y(T) = b_0$.

Where $f(t)$ isa source term A, B, C are constant coefficients and $Y(t)$ is known soluation which we want to determined. T is the constant representing the spin of inputs which the closed interval [0,T], a_0, b_0 are constants.

To obtained the numerical procedure by mean of shifted Legendre polynomials we assume that

(4.2) $D^2Y(t) = K_M^T\ \psi_M(t)$.

Where K_M is known unknown coefficient matrix which we need for approximation to calculate it. Appling fractional integral of order 2 and Lemma (2.2), we have

(4.3) $Y(t) = K_M^T\ G^2_{M\times M}\ \psi_M(t) + c_0t + c_1$.

By mean of boundary condition we have

$c_0 = a_0, \quad c_0 = \frac{b_0-a_0}{T} - \frac{1}{T}\ K_M^T\ G^2_{M\times M}\psi_M(t)\,|_{t=T}$.

Using the approximations

$a_0 + \frac{t(b_0-a_0)}{T} = F_M^T\ \psi_M(t)$, $\frac{t}{T}G^2_{M\times M}\psi_M(t)\,|_{t=T} = Q^{2,\varphi}_{M\times M}\psi_M(t)$.

Hence $Y(t) = K_M^T G^2_{M\times M}\psi_M(t) + a_0 + \frac{t(b_0-a_0)}{T} - \frac{t}{T}\ K_M^T\ G^2_{M\times M}\psi_M(t)\,|_{t=T}$

(4.4) $Y(t) = K_M^T G_{M\times M}^2 \psi_M(t) + F_M^T\, \psi_M(t) - K_M^T Q_{M\times M}^{2,\varphi} \psi_M(t) = K_M^T (G_{M\times M}^2 - Q_{M\times M}^{2,\varphi})\psi_M(t) + F_M^T\, \psi_M(t).$

Now

(4.5) $D^{\propto} Y(t) = D^{\propto}[K_M^T (G_{M\times M}^2 - Q_{M\times M}^{2,\varphi})\psi_M(t) + F_M^T\, \psi_M(t)].$

$= K_M^T (G_{M\times M}^2 - Q_{M\times M}^{2,\varphi}) H_{M\times M}^{\propto} \psi_M(t) + F_M^T\, H_{M\times M}^{\propto} \psi_M(t),$

and

(4.6) $f(t) = F_M^1\, \psi_M(t).$

Putting the equation (4.2), (4.3), (4.5) and (4.6)in equation (4.1), we have

(4.7) $AK_M^T\, \psi_M(t) + BK_M^T (G_{M\times M}^2 - Q_{M\times M}^{2,\varphi}) H_{M\times M}^{\propto} \psi_M(t) + BF_M^T\, G_{M\times M}^{\propto} \psi_M(t)$

$+CK_M^T (G_{M\times M}^2 - Q_{M\times M}^{2,\varphi})\psi_M(t) + CF_M^T\, \psi_M(t) = F_M^1\, \psi_M(t)$

$AK_M^T + K_M^T\, (G_{M\times M}^2 - Q^{2,\varphi})(BH_{M\times M}^{\propto} + CI) + BF_M^T H_{M\times M}^{\propto} + CF_M^T - F_M^1 = 0.$

Which is an algebriac type matrix equation which can easily be solved by Gaussian elimination method, using Matlab or Mathematica? Solving (4.7) for K_M^T and putting in (4.3) we get the required approximations for $Y(t)$.

Case II. If we consider (4.1) with the following boundary conditions

$$Y(0) = a_0, \qquad Y'(T) = b_1$$

Then by preceding at the same fashion in case I we get the required approximation equation as

(4.8) $AK_M^T + K_M^T\, (G_{M\times M}^2 - Q^{1,\varphi})(BH_{M\times M}^{\propto} + CI) + BF_M^T H_{M\times M}^{\propto} + CF_M^T - F_M^1 = 0.$

Where $F_M^T \varphi_M(t) = a_0 + tb_1,\ Q_{M\times M}^{1,\varphi} = tG_{M\times M}.$

Case III. If we consider (4.1) with following boundary conditions

$$Y(0) = a_0, \qquad Y'(0) = a_1$$

Then by preceding at the same fashion in Case I we get the required approximation equation as

(4.9) $AK_M^T + K_M^T\, G_{M\times M}^2 (BH_{M\times M}^{\propto} + CI) + BF_M^T H_{M\times M}^{\propto} + CF_M^T - F_M^1 = 0.$

Where $F_M^T \varphi_M(t) = a_0 + ta_1.$

Similarly the other cases can be easily discuss in same way.

On ward in this section, we provide some examples for different fractional order system represent by Bagley-Torvik equation. We also compare our results to that of other method.

Example 4.1. Consider

(4.10) $A\frac{d^2 Y(t)}{dt^2} + B\frac{d^{\propto} Y(t)}{dt^{\propto}} + CY(t) = 2 + \sqrt{\frac{t}{\pi}} + t^2, \qquad 1 < \propto < 2,\ 0 < t \le 1,$

Subject to the boundary conditions $y(0) = 0,\ y(1) = 1.$

Solution: we solve the Balgey-Tagley equation under the following values for the involved parameters $S = \{A = 1,\ B = 1,\ C = 1,\ \propto = 1.5\}$, and select source term for S as

$$f(t) = 2 + \sqrt{\frac{t}{\pi}} + t^2\ .$$

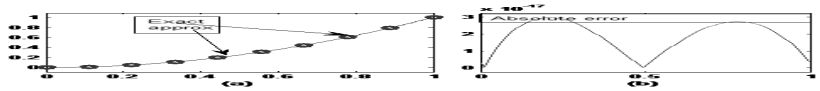


Fig.1. Comparison of exact and approximate solution of example 4.1, under the given parameters.

The exact solution of the above problem is $y(t) = t^2$. We solve this problem with the proposed method under the given value of parameters in *S*. The observation and simulation demonstrates that the solution obtained with the proposed method is highly accurate. The comparison of exact solution with approximate solution obtained using the given parameters is displayed in subplot (*a*) of (**Fig. 1),** while in subplot (*b*), we plot absolute deference $|Y_{approximate} - Y_{exact}|$ between the exact and approximate solution for scale level $M = 6$. We observe that our approximation has the best agreement to that of exact value.

Remark I: This problem was solved Raja *et.al* (2014) by evolutionary intelligence method for which the error was recorded , which was between 10^{-2} and 10^{-3} for scale level $M = 10$, while in our method for $M = 6$ error lies between 10^{-17} which is very small and gives information about the efficiency of our method.

Example 4.2. Consider

(4.11) $A\frac{d^2 Y(t)}{dt^2} + B\frac{d^{1.5} Y(t)}{dt^{1.5}} + CY(t) = f(t), \qquad 0 < t \le 1,$

Subject to the boundary condition $y(0) = 0,\ y(1) = 1.$

Where the source term $f(t) = \frac{38877t}{2125} - \frac{149369t^2}{4250} + \frac{26488t^3}{1275} - \frac{377t^4}{510} + \frac{13t^5}{51}$

(4.12) $+ \frac{45735911968175829\sqrt{t}(8000t^3 - 16240t^2 + 106240t - 2373)}{3349552222858064000} - \frac{339}{125}.$

And assume that the exact solution be

(4.13) $Y(t) = t^5 - \frac{29t^4}{10} + \frac{76t^3}{25} - \frac{339t^2}{250} + \frac{27t}{125}.$

Solving (4.10) by the proposed methods together with assuming the parameters value as

$$A = 1, B = \frac{18}{17}, C = \frac{13}{51}, \propto = 1.5.$$

The result are plotted in **(Fig.2)** where the subplot (*a*) shows the plot of approximate and exact values of Y (*t*) at scale level M = 6, while in subplot(b) of Figure(2), their absolute error has been plotted which is 1.2×10^{-16} for scale level M = 6. This problem was solved by Har-wavelet M.Rehman *et.al* (2011),where for scale level M = 8 the error was 2.5×10^{-7}, but by solving at our proposed scheme for scale level M = 8, we get error 1.5×10^{-17}. The very small error of our scheme for the aforesaid problem shows the efficiency of our proposed method.

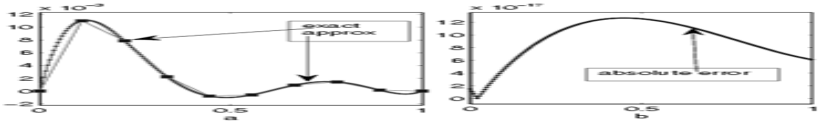


Fig.2. Comparison of exact and approximate solution of example 4.2, under the given parameters.

Example 4.3. $A\frac{d^2Y(t)}{dt^2} + B\frac{d^{1.5}Y(t)}{dt^{1.5}} + CY(t) = f(t), \quad 0 < t \leq 1,$ (4.14)

Subject to the boundary condition $y(0) = 1, \ y'(1) = 0.$

where the source term is $f(t) = \frac{t^4+24t^2+1}{2} + \frac{8t^2}{5}\sqrt{\frac{t}{\pi}}.$

Let the exact solution be $t^4 + 1$, and $A = 1, B = \frac{1}{2}, C = \frac{1}{2}, \propto = 1.5$

then solving (4.14) by our proposed method under the given values of parameters and under the given boundary conditions, we plot the approximate and exact value in **(Fig.3)** subplot (a) and their absolute error for scale level M = 6 in subplot (b). From the subplot (b) of (3), it is clear that the absolute error $|Y_{app} - Y_{exact}|$ lies bellow 10^{-18} at scale level M=6, however this error may be further decreased by enlarging the scale level. Thus our proposed method has the best agreement to that with exact value.

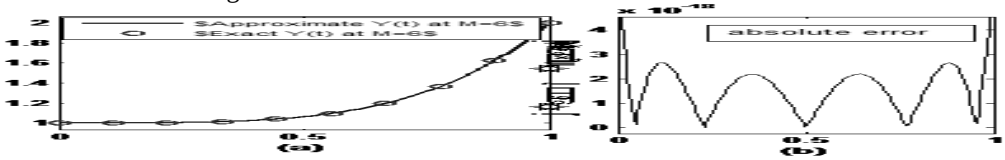


Fig.3. Comparison of exact and approximate solution of example 4.3, under the given parameters.

Example 4.4. . $A\frac{d^2Y(t)}{dt^2} + B\frac{d^{1.5}Y(t)}{dt^{1.5}} + CY(t) = f(t), \qquad 0 < t \leq 1,$ (4.15)

Subject to the boundary condition $y(0) = 1, \ y'(1) = 2.$

where the source term is $f(t) = 5e^{2t} + t^{1.5}E_{1,0.5}(2t)$

Let the exact solution be $Y(t) = e^{2t}$, and assuming that the involve parameters having the values

$A = 1, B = \frac{1}{2}, C = \frac{1}{2}, \propto = 1.5.$

then solving (4.15) by our proposed method under the given values of parameters and under the given boundary conditions. Where, we plot the approximate and exact value in Fig, 4 subplot (a) and in subplot (b) for scale level M=6 respectively. From the subplot (b) of **(Fig.4),** it is clear that absolute error $|Y_{app} - Y_{exact}|$ lies bellow 10^{-18} at scale level M=6, however this error may be further decreased by enlarging the scale level. Thus our proposed method has the best agreement to that of exact value. In next **(Table-1),** we give comparison of all examples at various scale levels .

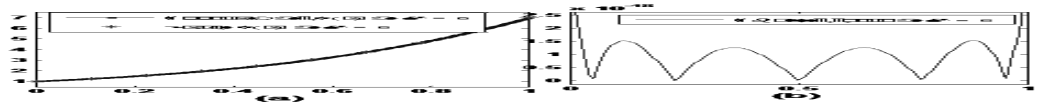


Fig. 4. Comparison of exact and approximate solution of example 4.4, under the given parameters.

Table 1. Maximum absolute error at various values of M for all four given example

M	Example 5.1	Example 5.4	Example 5.3	Example 5.4
4	1×10^{-7}	1×10^{-13}	1×10^{-10}	1×10^{-12}
5	1×10^{-10}	1×10^{-13}	1×10^{-16}	1×10^{-15}
6	1×10^{-17}	1×10^{-16}	1×10^{-17}	1×10^{-18}
7	1×10^{-17}	1×10^{-16}	1×10^{-17}	1×10^{-18}
8	1×10^{-18}	1×10^{-17}	1×10^{-18}	1×10^{-19}

From the table it is clear that as scale level increases the absolute error rapidly decreases, which indicates the validity of our proposed methods.

5. CONCLUSION AND FUTURE SCOPE

Above analysis and discussion take us to the conclusion that the proposed method is very efficient for the solution of initial/boundary value problems for fractional differential equations for the systems of Bagley -Torvik equations and other such like fractional differential equations. One can easily extended the method for the solution of other such type of problems with other kinds of boundary and initial conditions. Accuracy of this method is more than other numerical methods such as Legendre wavelet method, Adomain decomposition method, Homotopy perturbation and Variational iterations method etc. The new operational matrices obtained in this method can easily be extended to two dimensional and higher dimensional case which will help in the solution of fractional order partial differential equation. Our future work is related to extend the method to solve fractional order partial differential equation of higher dimension.

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