Sindh Univ. Res. Jour. (Sci. Ser.) Vol.46 (2): 99-102 (2014)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

A refinement of Heniz-logarithmic inequality for matrices

S. HUSSAIN[++], S. QAISAR*

Department of Mathematics, College of Science, Qassim University, P. O. Box 6644, Buraydah 51452, Saudi Arabia.

Received 12th June 2013 and Revised 9th March 2014

Abstract: In this note, we establish a refinement of Heinz-logarithmic inequality for unitarily invariant norms, which was obtained by Drissi in 2006. The obtained inequality gives a better estimates than already presented."

Keywords: Positive semidefinite matrix; convex function; unitarily invariant norms; matrix means.

1. INTRODUCTION

There are several means that interpolate the geometric and arithmetic means for two nonnegative real numbers, see [1, Chapter 4].(where is the chpt 4) Two well-known means for two scalars are the so-called Heinz mean $H_\nu(a,b)$ and the logarithmic mean $L(a,b)$, given by

$$H_\nu(a,b)=\frac{a^\nu b^{1-\nu}+a^{1-\nu}b^\nu}{2}\quad (a>0, b>0, 0\le\nu\le1),$$

$$L(a,b)=\frac{a-b}{\log a-\log b}=\int_0^1 a^t b^{1-t}dt\quad (a>0, b>0).$$

Note that $H_0(a,b)=(a+b)/2$ is the arithmetic mean and $H_{1/2}(a,b)=\sqrt{ab}$ is the geometric mean.

It is well known that the Heinz and logarithmic means interpolate between the geometric mean and the arithmetic mean:

$$\sqrt{ab}\le\frac{a^\nu b^{1-\nu}+a^{1-\nu}b^\nu}{2}\le\frac{a+b}{2},\quad (1.1)$$

$$\sqrt{ab}\le\int_0^1 a^t b^{1-t}dt\le\frac{a+b}{2}.\quad (1.2)$$

Let M_n be the space of $n\times n$ complex matrices and $\|\cdot\|$ stands for any unitarily invariant norm on M_n, i.e., $\|UAV\|=\|A\|$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$.

There are two celebrated inequalities for matrices related to the inequalities (1.1) and (1.2). Suppose that $A, B, X\in M_n$, and A, B are positive semidefinite. In 1993, Bhatia and Davis proved Bhatia, Davis. (1993) that if $0\le\nu\le1$, then

$$\left\|A^{1/2}XB^{1/2}\right\|\le\left\|\frac{A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu}{2}\right\|\le\left\|\frac{AX+XB}{2}\right\|.\quad (1.3)$$

In 1999, Hiai and Kosaki obtained the following inequality:

$$\left\|A^{1/2}XB^{1/2}\right\|\le\left\|\int_0^1 A^t XB^{1-t}dt\right\|\le\left\|\frac{AX+XB}{2}\right\|.\quad (1.4)$$

$$\left\|\frac{A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu}{2}\right\|\le\left\|\int_0^1 A^t XB^{1-t}dt\right\|,\quad (1.5)$$

where $\frac{1}{4}\le\nu\le\frac{3}{4}$.

The Heinz-logarithmic inequality (1.5) gives the relationship between the Heinz and logarithmic means for matrices, which extends the above results given by Bhatia-Davis and Hiai-Kosaki.

In this paper, We utilize the convexity of the function to obtain unitarily invariant norms inequality that leads to a refinement of the Heinz-logarithmic inequality (1.5).

2 MAIN RESULTS

To prove our main result, the following two Lemmas on convex function are needed:

Lemma 2.1. (Drissi. 2006) Let $A, B, X\in M_n$ such that A and B are positive semidefinite, then the function

$$\psi(\nu)=\frac{1}{2}\left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|\quad (2.1)$$

is convex on $[0,1]$.

[++]Corresponding Author: S. HUSSAIN Email: sabiriub@yahoo.com
*College of Mathematics and Statistics, Chongqing University, Chongqing,, Chongqing, 401331, P. R. China Email: shahidqaisar90@yahoo.com

Lemma 2.2 [6, 7]. If $f(x)$ is convex on $[a,b]$, then for $x \in (x_1, x_2) \subset [a,b]$

$$f(x) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}. \quad (2.2)$$

In what follows, we shall utilize the convexity of $\psi(v)$ in (1) to prove our result (Theorem 2.1) that is a refinement of the Heinz-logarithmic inequality (1.5).

Theorem 2.1. Let $A, B, X \in M_n$ such that A and B are positive semidefinite. If $\frac{1}{4} \le v \le \frac{3}{4}$, then

$$\begin{aligned}\left\| \frac{A^v X B^{1-v} + A^{1-v} X B^v}{2} \right\| &\le (4v_0 - 1)\left\| A^{1/2} X B^{1/2} \right\| + (1 - 2v_0)\left\| A^{1/4} X B^{3/4} + A^{3/4} X B^{1/4} \right\| \\ &\le \left\| \int_0^1 A^t X B^{1-t} dt \right\|, \qquad (2.3)\end{aligned}$$

where $v_0 = \min\{v, 1-v\}$. Note that when $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, the inequality (3) becomes equality.

Proof. If $\frac{1}{4} \le v \le \frac{1}{2}$, then by Lemma 2.1 and Lemma 2.2, we have

$$\psi(v) \le \frac{\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{4}\right)}{\frac{1}{2} - \frac{1}{4}} v - \frac{\frac{1}{4}\psi\left(\frac{1}{2}\right) - \frac{1}{2}\psi\left(\frac{1}{4}\right)}{\frac{1}{2} - \frac{1}{4}}.$$

That is,

$$\psi(v) \le (4v - 1)\psi\left(\frac{1}{2}\right) + (2 - 4v)\psi\left(\frac{1}{4}\right). \quad (2.4)$$

If $\frac{1}{2} \le v \le \frac{3}{4}$, then by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}\psi(v) &\le \frac{\psi\left(\frac{3}{4}\right) - \psi\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{2}} v - \frac{\frac{1}{2}\psi\left(\frac{3}{4}\right) - \frac{3}{4}\psi\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{2}} \\ &= (3 - 4v)\psi\left(\frac{1}{2}\right) + (4v - 2)\psi\left(\frac{3}{4}\right). \qquad (2.5)\end{aligned}$$

Since $\psi\left(\frac{3}{4}\right) = \psi\left(\frac{1}{4}\right)$, it follows from (5) that

$$\psi(v) \le (3 - 4v)\psi\left(\frac{1}{2}\right) + (4v - 2)\psi\left(\frac{1}{4}\right). \quad (2.6)$$

By (4) and (6), for $\frac{1}{4} \le v \le \frac{3}{4}$ we have

$$\psi(v) \le \begin{cases} (4v-1)\psi\left(\frac{1}{2}\right)+(2-4v)\psi\left(\frac{1}{4}\right), & if\ \frac{1}{4}\le v\le\frac{1}{2} \\ (3-4v)\psi\left(\frac{1}{2}\right)+(4v-2)\psi\left(\frac{1}{4}\right), & if\ \frac{1}{2}\le v\le\frac{3}{4}, \end{cases}$$

which implies

$$\psi(v) \le (4v_0-1)\psi\left(\frac{1}{2}\right)+2(1-2v_0)\psi\left(\frac{1}{4}\right) \tag{2.7}$$

with $v_0=\min\{v,1-v\}$.

Note that

$$\psi\left(\frac{1}{2}\right)=\left\|A^{1/2}XB^{1/2}\right\|, \quad \psi\left(\frac{1}{4}\right)=\frac{1}{2}\left\|A^{1/4}XB^{3/4}+A^{3/4}XB^{1/4}\right\|, \tag{2.8}$$

by (7) we have

$$\left\|\frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2}\right\| \le (4v_0-1)\left\|A^{1/2}XB^{1/2}\right\|+(1-2v_0)\left\|A^{1/4}XB^{3/4}+A^{3/4}XB^{1/4}\right\|.$$

which is the first inequality in (2.3). This proves the first part of (2.3).
On the other hand, by (2.8) and (1.5) we have

$$\psi\left(\frac{1}{2}\right)=\left\|A^{1/2}XB^{1/2}\right\| \le \left\|\int_0^1 A^{t}XB^{1-t}dt\right\|,$$

$$\psi\left(\frac{1}{4}\right)=\frac{1}{2}\left\|A^{1/4}XB^{3/4}+A^{3/4}XB^{1/4}\right\| \le \left\|\int_0^1 A^{t}XB^{1-t}dt\right\|.$$

So,

$$\begin{aligned}(4v_0-1)&\left\|A^{1/2}XB^{1/2}\right\|+(1-2v_0)\left\|A^{1/4}XB^{3/4}+A^{3/4}XB^{1/4}\right\| \\ &=(4v_0-1)\psi\left(\frac{1}{2}\right)+2(1-2v_0)\psi\left(\frac{1}{4}\right) \\ &\le (4v_0-1)\left\|\int_0^1 A^{t}XB^{1-t}dt\right\|+2(1-2v_0)\left\|\int_0^1 A^{t}XB^{1-t}dt\right\| \\ &\le \left\|\int_0^1 A^{t}XB^{1-t}dt\right\|,\end{aligned}$$

which is the second inequality in (3). The proof is completed.

3 Numerical Examples

In this section, we give two numerical examples to show that our result is better than the inequality (1.5).

Example 3.1. Let

$$A=\begin{pmatrix}10 & 0\\ 0 & 1\end{pmatrix}, X=B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \|\cdot\|=\|\cdot\|_2, v_0=\frac{1}{4}.$$

Then, by (1.55), we have

$$\left\|\int_0^1 A^{t}XB^{1-t}dt\right\|_2 = 4.0345.$$

By Theorem 2.1, we have

$$\left\| (4v_0 - 1) \left\| A^{1/2} X B^{1/2} \right\| + (1 - 2v_0) \left\| A^{1/4} X B^{3/4} + A^{3/4} X B^{1/4} \right\| \right\|_2 = 3.7008.$$

Example 3.2. Let

$$A = \begin{pmatrix} 1000 & 0 \\ 0 & 1 \end{pmatrix}, X = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \|\cdot\| = \|\cdot\|_2, v_0 = \frac{1}{4}$$

Then, by (1.5), we have

$$\left\| \int_0^1 A^t X B^{1-t} dt \right\|_2 = 144.6235.$$

By Theorem 2.1, we have

$$\left\| (4v_0 - 1) \left\| A^{1/2} X B^{1/2} \right\| + (1 - 2v_0) \left\| A^{1/4} X B^{3/4} + A^{3/4} X B^{1/4} \right\| \right\|_2 = 91.7255.$$

REFERENCES:

Bhatia. R. (2007) Positive Definite Matrices. Princeton University Press, Princeton,.

Bhatia, R., and C. Davis. (1993) More matrix forms of the arithmetic-geometric mean inequality. SIAM J. Matrix Anal. Appl. 14 132-136.

Bhatia, R. R. Sharma. (2011) Some inequalities for positive linear maps. Linear Algebra Appl. 436 1562-1571.

Bhatia R. (1997) Matrix Analysis. Springer-Verlag, New York.

Drissi, D. (2006) Sharp inequalities for some operator means. SIAM J. Matrix Anal. Appl. 28 (2006) 822-828.

Hiai, F., and H. Kosaki. (1999) Means for matrices and comparison of their norms. Indiana Univ Math. J. 48 899-936.

Wang, S., L. Zou, Y. Jiang. (2012) Some inequalities for unitarily invariant norms of matrices. J. Inequal. Appl. 2011.