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FEM based Numerical Approximation Model for the Ambrosio–Tortorelli Energy Model and the Numerical Simulation of the Edge Detection and Image Denoising

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Abstract: We propose a FEM (Finite Element Method) based semi implicit simulation scheme for the Ambrosio–Tortorelli energy functional for the denoising and segmentation of noisy images. Our main goal is to detect the corners in noisy as well as noise free images. The fast converging semi implicit iterative numerical scheme is derived for the numerical solution of nonlinear partial differential equation which is obtained from the minimization of given energy functional using direct methods from the calculus of variations. The scaling parameters are used as regularization weights; these local adaptive smoothness parameters generally play a crucial role as regularization devices in the variational techniques for the image processing and computer vision research.

Keywords: Image Denoising, Image Segmentation, Variational method, Finite element Methods.

1. INTRODUCTION

The mathematical Optimization is a modern approach and plays the crucial role in the area of image analysis and computer vision research. The fundamental idea behind the variational methods is the minimization of an energy functional which generally involves the features of the images. The methods based on the "minimization of energy functionals" have attracted many researchers from the mathematical community from all around the world to contribute in the solutions of image processing and computer vision problems. As these variational methods are based on the idea of regularization which was initially more or less started by Tikhonov (1963).

Though the regularization approaches are very famous and suitable for the image processing and computer vision problems, as suggested in the literature Amur (2011), Amur (2012), Amur (2013), Amur *et al* (2013), Amur *et al.,* (2014), Schnörr (1994), Rudin (1992), Zhai (2011) , Engl *et el.,* (1996), Otmar Scherzer and Weikert (2000), Strong(1997), Perona and Malik (1990), Mumford and Shah(1985), Tikhonov and Arsenin (1977)] but the ambiguity in the computation of the accurate and efficient diffusion at all locations of the image is still a challenging task for the image researchers. Many algorithms and computational strategies have been proposed to solve these regularization problems but the accuracy is still the open question for the numerical analysis practitioners.

The computational approaches based on the finite element methods for the mathematical image processing problems are almost new ideas of approximations for these problems Amur (2011), Amur (1912), (Amur 3013), Amur *et al.,* 2013), Amur, *et al.,* 2014) . These strategies are more accurate and efficient as compared to the usual strategies which use the finite difference methods for the solution of PDEs (Partial Differential Equations) in the image processing but the speed of convergence in the FEM based approaches is still a problem which is solvable using the usual computational speed optimization strategies by modern computer programmers. (Chandio *et al.,* 2013) In such like situations of slow convergence the modern computer languages like FreeFem++ Hecht (2013) provide a robust platform for FEM users to design such like efficient algorithms which can facilitate the PDEs based image practitioners to achieve their goals like the solution of speed convergence problem in FEM based methods. For example recently the novel adaptive strategies based on the mesh adaption for image motion problems were proposed in Amur (2011), Amur (2012), Amur (2013), Belhachmi and Hecht (2011) .

Identifying the discontinuities in the form of curves, line segments in the images is a classical problem called edge detection. One is therefore interested to identify those regions in image where brightness varies sharply. Many methods are suggested in literature Zhai *et el.,* (2011), Marr. and Hildreth (9180), Rohr (1994). In this paper our goal is to solve the optimization problem called Ambrosio–Tortorelli energy functional using P_1 finite element method to determine simultaneously the diffused image and segmented image from a given noisy image. We design a semi implicit discrete approximation model for the solution of nonlinear partial differential equation obtained from the minimization of selected energy functional to achieve our goal.

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2. PROBLEM DESCRIPTION

A variational method is considered for the image reconstruction of an original u image and the edge function v from a given noisy image f on the open and bounded two dimensional domain Ω. We assume that the image f is generated from the image u by adding the noise η. We consider the minimization of the following energy functional.

$$E(u,v)=\int_{\Omega}(\alpha\ (\mathrm{c}|\nabla v|^2+\frac{(1-\mathrm{v})^2}{4c})+\mathrm{v}^2|\nabla u|^2+\beta(\mathrm{u}-f)^2)\,\mathrm{dx}\ldots.(1)$$

The smoothness weights α, c and β, are strictly positive uniform parameters on the domain Ω, $x=(x_1,x_2)^T\in\Omega$, $\nabla=(\partial x_1,\partial x_2)^T$ denotes the gradient operator and $|\cdot|$ denotes the Euclidean norm. u Denotes the diffusion image and v denotes the edge function. As the given energy functional in equation (1) $E(\mathrm{u},\mathrm{v})$ is not convex jointly both in the arguments u and v therefore unique local minima is difficult to obtain and the gradient system associated to the Euler-Lagrange equation may get stuck in the local minima.

Applying the direct results from the calculus of variations the energy functional (1) satisfies the following coupled system of Euler-Lagrange equations

$$-\mathrm{div}(\mathrm{v}^2\,\nabla\mathrm{u})+\beta(\mathrm{u}-f)=0,\ in\ \Omega\ldots\ldots..(2)$$

$$-c\,\Delta v-\frac{(1-\mathrm{v})}{4c}-\frac{v}{\alpha}|\nabla u|^2=0,\ in\ \Omega\ldots\ldots(3)$$

Where $\frac{\partial u}{\partial n}=0,$ and $\frac{\partial v}{\partial n}=0\,on\ \partial\Omega$, one is therefore looking for a steady state solution of the system

$$-\mathrm{div}(\mathrm{v}^2\,\nabla\mathrm{u})+\beta(\mathrm{u}-f)=\frac{\partial u}{\partial t},\ in\ \Omega\times(0,T]\ldots..(4)$$

$$-c\,\Delta v-\frac{(1-\mathrm{v})}{4c}-\frac{v}{\alpha}|\nabla u|^2=\frac{\partial u}{\partial t},\ in\ \Omega\times(0,T]\ldots(5)$$

For our simplicity we assume that $\Delta\mathrm{v}=0$ in equation (3), this assumption allows us to solve this system directly for v and yields

$$-\mathrm{div}(\frac{1}{\left(1+\frac{4c}{\alpha}|\nabla u|^2\right)^2}\nabla\mathrm{u})+\beta(\mathrm{u}-f)=0\ldots(6)$$

$$v=\frac{1}{1+\frac{4c}{\alpha}|\nabla u|^2}\ldots\ldots..(7)$$

Weak formula for (6) is denoted as

$$b(u,w)=l(f,w)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(8)$$

where we are interested to compute $u\in H^1(\Omega)$ where

$$\begin{cases} b(u,w)=\int\limits_{\Omega}\left(\dfrac{\alpha\ \nabla u\cdot\nabla w}{(1-\frac{4c}{\alpha}|\nabla u|^2)^2}\right)dx+\beta\int\limits_{\Omega}(u\cdot w)\ dx \\ l(f,w)=\int\limits_{\Omega}f\cdot w\ dx\ \ \forall\ w\in H^1\ \ldots\ldots\ldots..(9)\end{cases}$$

For the details on the theory of weak formulation, Finite elements and Sobolev spaces we refer the reader to review the book (Brenner S. C and L. R. Scott (1994)).

3. NUMERICAL SCHEME

3.1 Discrete formulation:

The discrete formulation of the problem (8) and (9) is given as

$$b(u_h,\mathrm{w}_h)=l(f,\mathrm{w}_h)\ ,\quad\forall\ w_h\in X_h\ldots\ldots\ldots\ldots.(10)$$

Where

$$X_h:=\left\{\ w_h\in C^0(\overline{\Omega})\middle|\ \forall K\in T_h,\ w_h|_K\in P_1(K)\right\}$$

Here X_h is the discrete space with $P_1(\mathrm{K})$ finite elements. The computational domain is considered as triangular grid T_h with maximum size of each element $h>0$. We have

$$\begin{cases} b(u_h,w_h)=\int\limits_{\Omega}\dfrac{\alpha\ \nabla u_h\cdot\nabla w_h}{(1-\frac{4c}{\alpha}|\nabla u_h|^2)^2}dx \\ \qquad\qquad+\beta\int\limits_{\Omega}(u_h\cdot w_h)\ dx \\ l(f,w_h)=\int\limits_{\Omega}f\cdot w_h\ dx\ \ \forall\ w_h\in H^1 \\ v_h=\dfrac{1}{1+\frac{4c}{\alpha}|\nabla u_h|^2}\end{cases}\ldots..(11)$$

It is not very straight forward to solve the above problem (11) because of the nonlinearity in the argument "u", therefore the following semi implicit numerical scheme is proposed for the steady state solution $(t \to \infty)$ of the problem (4) where the time derivative is discretized using the forward difference operator.

$$(I - \tau A_{\alpha,\beta})U^{K+1} = U^K + \tau L ....(12).$$

This numerical scheme is derived from the discrete version of the weak formulation associated to the problem (4) along with the substitution of edge function v.

$$\begin{cases} b(u^{k+1}{}_h, w_h) = \tau \int\limits_\Omega \dfrac{\alpha \ \nabla u^{k+1}{}_h \cdot \nabla w_h}{(1 - \frac{4c}{\alpha}\left|\nabla u^k{}_h\right|^2)^2} dx \\ \qquad + \tau \cdot \beta \int\limits_\Omega (u^{k+1}{}_h \cdot w_h) \ dx ....(13) \\ l(f, w_h) = u^k \cdot w + \tau \cdot \beta \int\limits_\Omega f \cdot w_h \ dx \ \ \forall \ w_h \in H^1 \\ v_h = \dfrac{1}{1 + \frac{4c}{\alpha}\left|\nabla u_h\right|^2} \end{cases}$$

$$U = \left[u_1, u_2, ..., u_N\right] ....(14)$$

Where $u_1, u_2, ..., u_N$ diffusion at N number of nodes vector L is obtained from the linear form $l(f, w_h)$ given in the equation (9). The index k shows the iterative procedure and τ denotes the time step. The numerical approximation model equation (12) is unconditionally stable because the bilinear form given in equation (13) is symmetric.

4. RESULTS AND DISCUSSION

We have successfully applied the two iterations of the given fast converging and efficient numerical approximation model (12) which based on FEM based discretization with triangular grid as the domain of computation. The computational procedure takes less than a second to complete two iterations for the solution of given 256 by 256 image matrix. The computational algorithm was tested in FreeFem++ [Hecht (2013)]. In this paper we have used the following combination of regularization parameters given in the discrete problem.

α	β	c
0.001	0.01	5

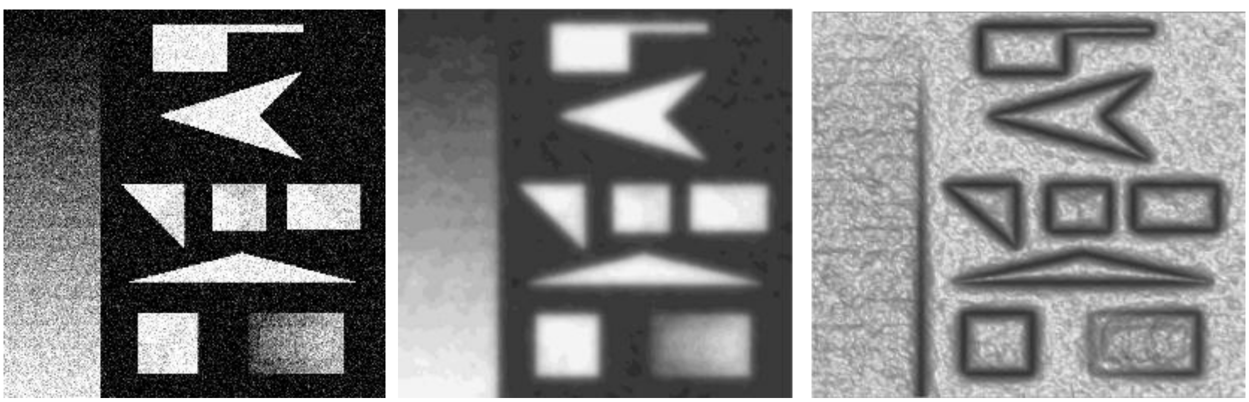
Efficiency of the algorithm is very clear from our simulation results as given in the **(Fig.1. (A-C), Fig.2.(D-F) and Fig. 3.(G-I)).**

Example 1. In this example we consider the famous test image called camera man downloaded from http://graphics.cs.williams.edu/data/images. Gaussian noise is added in the image with standard deviation (SD) =20, the image is successfully denoised **(Fig. 1. (B)** but due to the blurring effects some useful information in background of the camera man image is slightly lost. The robust effects of Perona-Malik diffusitivity on the edge information were observed in the form of sharpness in corners, for more details about the control of Perna-Malik diffusion on edges we refer the reader to our previous work Amur (2012). The segmentation of the given noisy image is achieved successfully as shown in **(Fig..1(C).** All the corners of the foreground image the Cameraman and camera stand along with its back ground edges are segmented successfully as shown in **(Fig.. 1(C).** In the observed segmentation of the given noisy image, the edges are appeared as darker corners whereas the regularized regions are brighter. All the experimental results of this example confirm the effectiveness of the given Numerical scheme.

Example.2 In this example we apply our algorithm on the image with combinations of the shapes like triangles and rectangles which was downloaded from http://users.fmrib.ox.ac.uk/ . The results for this case are given in **(Fig. 2. (D-F).** The **(Fig..2 (E)** shows denoising of the image and **(Fig..2 (F)** shows the segmentation. In the first case we have denoised and segmented the noisy image where the successful diffusion along with all the necessary singularities in the noisy image are recovered with no false appearance of any edge in the given image. This is the fulfillment of our main goal. The edges around and inside the white regions are efficiently detected and no false edges were observed. Thus it is very clear that the edge function copes very well with brightness changes. In the second case of this example we have provided the better quality results for the segmentation of original image **(Fig. 3. (G-I).** In this case no noise was added in the original image and we tested our algorithm for the segmentation operation. In this case the smoothed image is obtained along with successfully segmented image with better quality as compared to the previous case of noisy images. It is very clear from the obtained results that our method can perform very well in the presence of noise as well as without added noise and can segment the images along with fine structures. The denoising process for the camera man image Figure. 1(A-C) with added Gaussian noise (with SD=20) was observed successful but some brightness variations were observed in the background of the denoised image which is still a question in this task. One cannot say something about these brightness changes that those are because of ill-posedness of the problem but may be because of one of the the drawbacks of the variational techniques as successful computations of the grey values at all locations specifically when the images are given with complex structures in their back grounds. The diffused images in the second image **(Fig..2 (D-F)** were observed with good quality and the given method performs very well even in the case when the noise scale was increased (with SD=40).

(A) (B) (C)

Fig. 1. (A) Camera man Image with added noise SD=20 (B) Denoising with given numerical approximation model (C) Image Segmentation of given noisy image (Edge function v).



(D) (E) (F)

Fig. 2. (D) Image with angles and corners with added noise SD=40 (E) Denoising with given numerical approximation Model (F) Image Segmentation of given noisy image (Edge function v).

(G) (H)

Fig. 3. (G) Original Image with angles and corners (H) Smoothed image using given approximation model (I) Image Segmentation of original image without noise (Edge function v).

5. CONCLUSION

A robust computational framework was proposed for the Ambrosio–Tortorelli variational model in this paper for the image denoising and the image segmentation. Generally the iterative numerical procedures like the semi implicit numerical scheme designed in this paper are the suitable FEM based numerical strategies for the solution of nonlinear PDEs like (4) and (6). The proposed numerical iterative strategy was framed in such a way that initialization of numerical solution was kept automatic. The method given in this paper performs reasonably well and produces robust simulations for image segmentation as well as image denoising. The given numerical algorithm converges very fast and very satisfactory results were observed as shown in Fig. (1-3). Three experiments were performed on different images. In the first case the experiments were performed with the noisy images with different noise scales and the last the experiment was performed using the image without added noise, this was only to check the performance of the given method for the better segmentation of the images. In this case the method outer performs as compared with the case of noisy images. In all three cases we have denoised and segmented the images with successful diffusion and all the necessary singularities in the noisy image were recovered with no false appearance of any edge. In the case of noise free image the quality of image segmentation was improved as compared to the previous case of noisy images.

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