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On Local Error Bound of a Modified Ordinary Differential Equation Solver

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Abstract: In this paper, a second order accurate numerical method for the solution of Ordinary Differential Equations (ODEs) referred to as the Modified ODE Solver (Memon *et al.*, 2014) is analyzed on the basis of bounds on local truncation error and step size. The error and step size bounds for the modified ODE solver have been formulated by following the approach of Lotkin (1951) and are compared to those of second order accurate Improved Euler (IE) and Improved Modified Euler (IME) methods. The Modified ODE Solver is found to have the minimum local error bound and the maximum step size bound with the help of supporting numerical examples.

Keywords: Ordinary Differential Equation, Numerical Solution, Local truncation error, error bound, step size bound,

1. INTRODUCTION

Differential equations play a vital role in scientific and engineering fields. These equations often arise as a result of mathematical formulation of various physical problems such as, mechanical systems, electrical networks, fluid flows, population dynamics, radioactive decay, rate of chemical reactions, etc. (Boyce and DiPrima, 2001). There is no general analytical method to solve every differential equation. A number of analytical methods exist for finding an exact solution to an ordinary differential equation. However, the limitation of analytical techniques to solve the nonlinear differential equations has impeded the use of numerical methods for obtaining an approximate solution of the problem (Garewal, 2000). Numerical methods for initial value problems of ordinary differential equations are efficient mathematical tools that have made an enormous progress in the context of modern computers. Mathematicians have devoted themselves to continue research on methods for the numerical integration of initial value problems in ordinary differential equations. Some of them have devised methods as an improvement over the conventional methods, such as the trapezoidal method, improved Euler method, modified Euler method, Runge-Kutta and multistep methods. See for example (Gourlay, 1970; Villatro, 2004; Selvakumar, 2012; Abraham, 2007; Akanbi, 2010a; Chandio and Memon, 2010; Evans, 1989; Wazwaz, 1990; Rabiei and Ismail, 2011; Qureshi *et al.*, 2013). Others have introduced new methods, such as methods based on exponential sums, trigonometric interpolants, rational interpolants, and extrapolation methods, refer for example (Glaser and Rokhlin, 2009; Fatunla, 1976; Ogunride and Fadugba, 2012; Sunday and Odekunle, 2012; Momodu and Aashikpelokhai, 2008; Pandey, 2012; Usman and Mukhtar, 2013), thus, resulting in an immense literature offering variety of Ordinary Differential Equation (ODE) solvers.

The Modified ODE Solver developed by Memon *et al*. (2014) is given as:

$$\left.\begin{aligned} & y_n = y_{n-1} + \frac{1}{2}\left(k_1 + k_2 \right) \\ & \text{where} \\ & k_1 = \Delta t\, g\left(t_{n-1}, \mathrm{y}_{n-1}\right) = \Delta t\, g \\ & \tilde{k} = \Delta t\, g\left(t_{n-1} + \frac{\Delta t}{2}, \mathrm{y}_{n-1} + \frac{k_1}{2}\right) \\ & k_2 = \Delta t\, g\left(t_{n-1} + \Delta t, \mathrm{y}_{n-1} + \tilde{k}\right) \end{aligned}\right\} \qquad (1)$$

In the present paper, the method given by Eq. (1) is investigated for local error bound and step size bound-characteristics crucial to the overall efficacy of a numerical method.

2. MATERIALS AND METHODS

Definition 1: The local truncation error τ_{n+1} at $t = t_{n+1}$ of an explicit single-step method $y_{n+1} = y_n + \Delta t\ \psi\left(t_n, y_n ; \Delta t\right)$ is given as:

$$\tau_{n+1} = y\left(t_{n+1}\right) - y_{n+1},$$

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with the assumption that no previous error has been made, that is, $y(t_n)=y_n$.

Definition 2: The local error τ_{n+1} of a numerical method of order p is given as:

$$\tau_{n+1}=\phi(t_n,y_n)(\Delta t)^{p+1}+O(\Delta t)^{p+2}$$

where the function $\phi(t_n,y_n)$ is known as the principal error function, whereas, $\phi(t_n,y_n)(\Delta t)^{p+1}$ is known as the principal local truncation error.

The Taylor series expansion for the exact solution $y(t_{n+1})$ at $t=t_{n+1}$ is given as,

$$y(t_{n+1})=y(t_n)+\Delta t\,y'(t_n)+\frac{(\Delta t)^2}{2}y''(t_n)+\frac{(\Delta t)^3}{6}y'''(t_n)+O(\Delta t)^4$$

$$=y(t_n)+\Delta t g(t_n,y(t_n))+\frac{(\Delta t)^2}{2}g'(t_n,y(t_n))$$

$$+\frac{(\Delta t)^3}{6}g''(t_n,y(t_n))+O(\Delta t)^4 \qquad (2)$$

where $g'=g_t+g\,g_y$

$$g''=g_{tt}+2gg_{ty}+g^2g_{yy}+g_tg_y+gg_y{}^2$$

Similarly, the Taylor series expansion of the Modified ODE Solver is given as,

$$y_{n+1}=y_n+\Delta t\,g+\frac{(\Delta t)^2}{2}g'+\frac{(\Delta t)^3}{4}g''+O(\Delta t)^4$$

$$=y_n+\Delta t\,g(t_n,y_n)+\frac{(\Delta t)^2}{2}g'(t_n,y_n)+\frac{(\Delta t)^3}{4}g''(t_n,y_n)+O(\Delta t)^4 \qquad (3)$$

Assuming $y(t_n)=y_n$, the local truncation error of the Modified ODE Solver is

$$\tau_{n+1}=y(t_{n+1})-y_{n+1}=\frac{-(\Delta t)^3}{12}g''+O(\Delta t)^4 \quad (4)$$

Thus, the principal error function for the Modified ODE Solver is given as

$$\phi(t_n,y_n)=\frac{-1}{12}g''=\frac{-1}{12}\left(g_{tt}+2gg_{ty}+g^2g_{yy}+g_tg_y+gg_y{}^2\right) \quad (5)$$

Lotkin (1951) proposed following bounds known as Lotkin's bounds for the function g and its partial derivatives for $t\in[a,b]$ and $y\in(-\infty,\infty)$:

$$|g(t,y)|<M \quad \left|\frac{\partial^{i+j}g}{\partial t^i\partial y^j}\right|<\frac{N^{i+j}}{M^{j-1}}, \qquad i+j\le p$$

where M and N are positive constants and p is the order of accuracy of the numerical method. Using Lotkin's bounds,

$$|\phi(t_n,y_n)|<\frac{1}{2}(\Delta t)^3MN^2$$

And thus a bound for the principal local error of the Modified ODE Solver is obtained as:

$$\left|\phi(t_n,y_n)(\Delta t)^3\right|<\frac{1}{2}(\Delta t)^3MN^2 \qquad (6)$$

For an explicit single-step method, the bound on principal local error is also a bound for its local error. In view of this fact, the local error bound for the Modified ODE Solver is obtained as:

$$|\tau_{n+1}|<C(\Delta t)^3, \qquad C=\frac{1}{2}MN^2 \qquad (7)$$

(Table 1) presents the local error bounds and the step size bounds for Improved Euler, Improved Modified Euler and Modified ODE Solver.

Table 1 Comparison of error and step size bounds

Method	**Local error bound**	**Step size bound**
Improved Euler (IE) (Runge, 1895)	$\frac{2}{3}MN^2(\Delta t)^3$	$\left(\frac{1.5\times Tol}{MN^2}\right)^{1/3}$
Improved Modified Euler (IME) (Abraham, 2007)	$\frac{5}{6}MN^2(\Delta t)^3$	$\left(\frac{1.2\times Tol}{MN^2}\right)^{1/3}$
Modified ODE Solver (Memon *et al*., 2014)	$\frac{1}{2}MN^2(\Delta t)^3$	$\left(\frac{2\times Tol}{MN^2}\right)^{1/3}$

Where Tol denotes the tolerance on the local truncation error.

3. RESULTS AND DISCUSSION

In order to numerically investigate the error and step size bounds shown in table 1, consider the examples given below.

Example 01 (Akanbi, 2010b) Consider the initial value problem $y'=y,\ y(0)=1$ where $t\in[0\ \ 1]$. The exact solution is $y(t)=\exp(t)$.Now, for the problem under consideration, using Lotkin's bounds, we can take $M=3$ and $N=1$.

(Fig. 1) shows a comparison between the local error bounds of the Improved Euler (IE), Improved Modified Euler (IME) and the Modified ODE Solver for different values of the step size. It can be seen form the figure that for each choice of the step size, the local error of the Modified ODE Solver possesses the minimum bound in comparison to that of both the Improved Euler (IE) and Improved Modified Euler (IME) methods. The smaller bound on the local truncation error of the Modified ODE Solver makes it feasible to work with relatively larger step size as shown in **(Fig. 2)**, where the bounds on step size are obtained for IE, IME and Modified ODE Solver, corresponding to the different values of local error tolerance

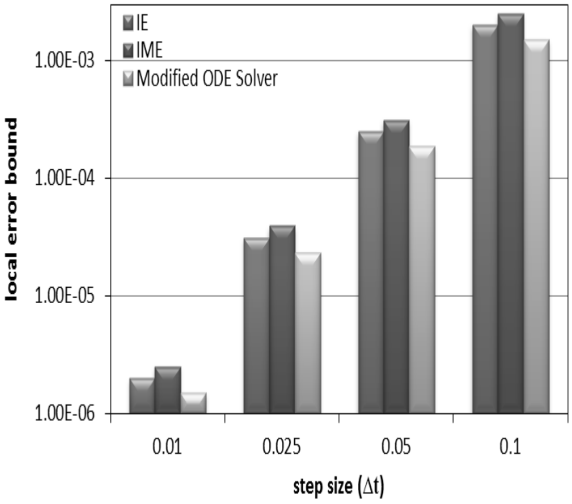

Fig. 1. Local error bound versus step size for example 01

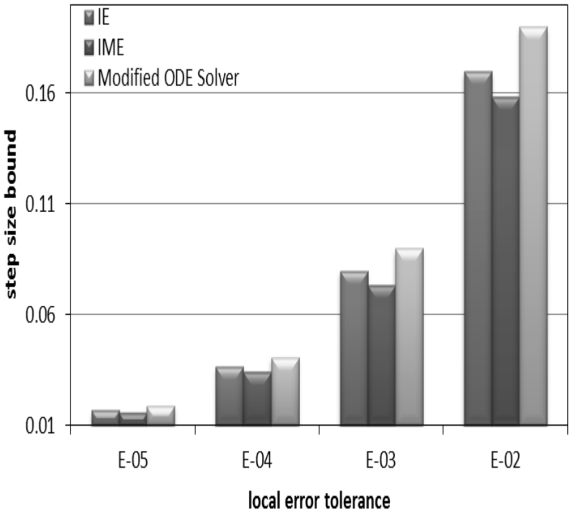

Fig. 2. Step size bound versus local error tolerance for example 01

Example 02 (Lotkin, 1951) Consider the initial value problem $y' = t + y;\ y(0) = 0$ where $t \in [0 \quad 1]$. The exact solution is $y(t) = e^t - t - 1$. For this example, we may take $M = 2$ and $N = 1$ using Lotkin's bounds.

(Fig. 3 and 4), respectively, present visualization to the bounds on local error and step size for IE, IME and Modified ODE Solver for the initial value problem in example 02. As evident from these figures, the Modified ODE Solver is better in respect of the local error bound as well as the step size bound than its counterparts IE and IME methods. In fig. 3, bounds on the local error are obtained taking different step sizes; the Modified ODE Solver possesses the smallest error bound. Similarly, for a given local error tolerance, the Modified ODE Solver can take into account a larger step size in comparison to both IE and IME methods as shown in **(Fig. 4)**.

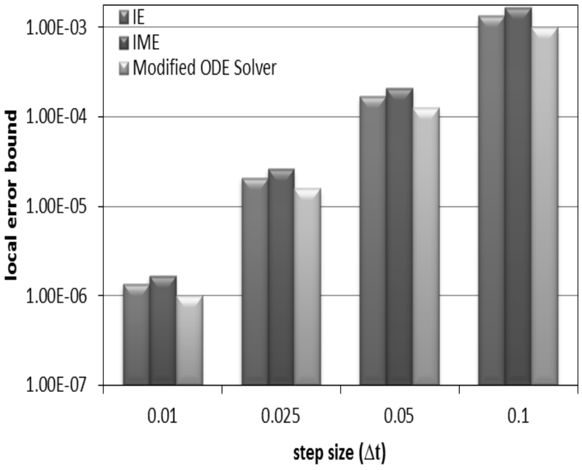

Fig. 3. Local error bound versus step size for example 02

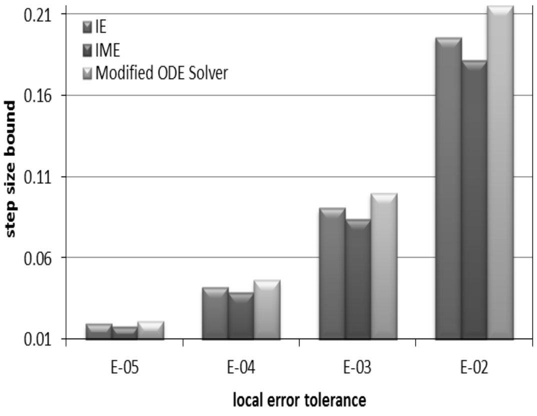

Fig. 4. Step size bound versus local error tolerance for example 02

Example 03 (Ralston, 1962) Consider the nonlinear initial value problem $y' = 1 - y^2;\ y(0) = 0$ where $t \in [0\ \ 1]$. The exact solution is $y(t) = \tanh t$. For this example, we have $M = 1$ and $N = 2$ using Lotkin's bounds.

Figs. 5 and 6 respectively, provide comparison of local error and step size bounds of IE, IME and Modified ODE Solver for the nonlinear initial value problem in example 03. It can be observed from these figures, the Modified ODE Solver is better in respect of the local error bound as well as the step size bound in comparison to both IE and IME methods. In **(Fig 5),** the local error bounds are obtained corresponding to the different step sizes; the Modified ODE Solver possessing the minimum error bound. Furthermore, for a given local error tolerance, the Modified ODE Solver takes into account a larger step size than both IE and IME methods as illustrated by **(Fig 6),**

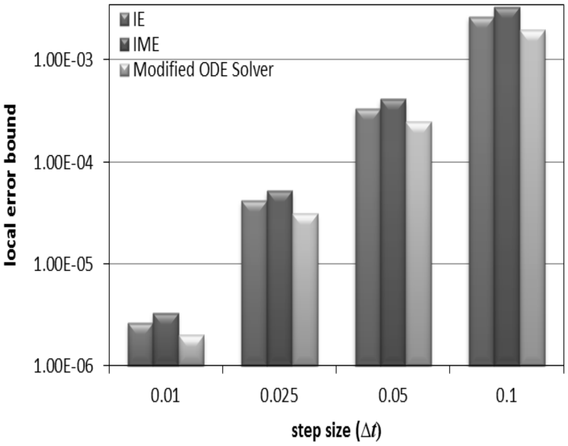

Fig. 5. Local error bound versus

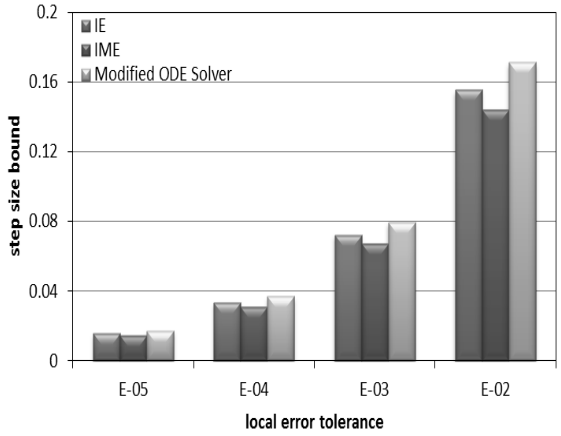

Fig. 6. Step size bound versus local error tolerance for example 03

4. CONCLUSION

In the present work, the attention was focused on the analysis of local error bound and step size bound of a Modified ODE Solver. The error and step size bounds of the Modified ODE Solver have been obtained using Lotkin's approach. The resulting bounds are compared to those of two well-known second order accurate ODE solvers taken from literature. These include Improved Euler (IE) and Improved Modified Euler (IME) methods. The Modified ODE Solver is found to possess the minimum error bound and the maximum step size bound with the help of supporting numerical examples, thus making it more efficient than its counterparts.

5. FUTURE WORK

Future work may be devoted to the derivation, analysis and comparison of the global error bound of the Modified ODE Solver with that of Improved Euler (IE) and Improved Modified Euler (IME) methods.

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