Sindh Univ. Res. Jour. (Sci. Ser.) Vol.46 (3) 373-376 (2014)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

Starting flows for a fractional Oldroyd-B fluid between two coaxial cylinders

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Received 1st June 2014 and Revised 15th August 2014

Abstract: In In this article we have discussed some starting flows for fractional Oldroyd-B fluid between two coaxial cylinders. Firstly we have model the governing equation for unsteady rotational flow of the incompressible fluid of a generalized Oldroyd-B fluid in an annular pipe. And then we have formulate the two physical fluid flow problem between two coaxial cylinders namely; *(i)* the outer cylinder generates the effective oscillation in terms of harmony *(ii)* an the abrupt rotation of the outer cylinder with consistent speed. While the inner cylinder kept fixed for both the situation. Exact analytical solutions for these two types of flows are obtained in term of Mittag-Leffler function by using Webers' and Laplace transforms for sequential fractional derivative. Moreover, the limiting cases of our solution correspond to the same solutions for Maxwell, Second grade and Newtonian fluid models performing the same motion.

Keywords: Exact Solution, Oldroyd-B Fluid, Fractional Derivative, Coaxial Cylinders, Harmonic Oscillation.

1. INTRODUCTION

In the been classified is the methodology given by (Rivilin, *et al.*, 1955), (Truesdell, Dunn *et al.*, 1992) (Rajagopal, 1993), who present constitutive relations for the stress tensor literature, the engineers, physicist, and mathematicians are presented with the challenge from the non- non-linear mechanics, as the non-linearity demonstrates itself in various manners.

One of simplest way in which the viscoelastic fluids have as a function of the symmetric part of the velocity gradient and its higher derivatives. There are many fluids such as second grade, Maxwell, Oldroyd-B etc, which lies in the category of viscoelastic fluid and form the subclass of viscoelastic fluid. In order to describe the viscoelasticity (Rossihin *et al.*, 2001) the fractional calculus approach is very important. Recently, fractional calculus approach has encountered much success in the description of complex dynamics. In particular, it has been proved to be valuable tool to handle viscoelastic properties. Viscoelasicity is the prime differential equation of fractional derivative model, and it is replaced by the order of the integer's Riemann-Liouville fractional calculus operator. In this manner we could define integer order derivatives (Podlubny, *et al.*, 1999).

Motion of the fluidity in the rotating boy's neighborhood is always creates a kind of interest to the Industry as well Academics. The cylindrical rotation has the initialization from the starts, and it has been utilized industry, and of course it has the very interesting applications in this manner. And it has been extensively in the food researched by the (Taylor, 1923). so the practical importance of viscoelasticity, (Tong and Liu 2005) studied the unsteady rotational flow of a viscoelastic fluid in an annular pipe. An exact solution for the flow of viscoelastic fluid with fractional Oldroyd-B fluid in an annular pipe was studied by (Tong, 2005). Exact solution on unsteady couette flow of generalized Maxwell fluid with fractional derivative was studied by (Shaowei, *et al.*, 2006) Rotating flows of viscoelastic fluid with fractional Maxwell model between coaxial cylinders are studied by (Qi and Ji) and exact solutions of unsteady couette flow of generalized second grade fluid are studied by (Tan *et al.)* Moreover, many researchers (Tan *et al.*, Xu, *et al.*, 2002) are engaged in obtaining the exact solutions for different fluid flow problems for viscoelastic fluid with fractional derivative.

In the present analysis the fractional calculus approach is used in the constitutive relationship of an Oldroyd-B fluid model. The flow of viscoelastic fluid between two coaxial cylinders is analyzed. Weber an Laplace transforms are used to reach at the proper solutions for the velocity fields, for the fractional derivative for two types of flow, namely, *(i)* Harmonic oscillation by the outer cylinder, and *(ii)* the constant rotation of the outer cylinder with speed of $\_U_0$ by keeping inner cylinder fixed for both the cases. Moreover, the limiting cases of our solution correspond to similar solution for fractional Maxwell, fractional second grade and Newtonian fluid performing the same motion.

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2. MATHEMATICAL FORMULATION OF THE PROBLEM

An incompressible generalized Oldroyd-B fluid model has the following the constitutive relationships

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \left(1+\lambda^{\alpha} D_t^{\alpha}\right)\mathbf{S} = \left(1+\theta^{\beta} D_t^{\beta}\right)\mathbf{A}_1 \tag{1}$$

where $\mathbf{T}$ the Cauchy stress tensor, $-p\mathbf{I}$ is the spherical stress, $\mathbf{S}$ the extra stress tensor, $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathbf{T}}$ is the first Rivlin-Ericksen tensor and $\mathbf{L}$ is the velocity gradient, μ the dynamic viscosity, λ and θ are relaxation and retardation times, respectively, α and β are fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$ and D_t^{α} and D_t^{β} are the fractional differentiation operators of order α and β with respect to t, respectively and may be defined as [7, 8]

$$D_t^{p} f(t) = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{p}}\, d\tau, \quad 0 \le p \le 1, \tag{2}$$

in where $\Gamma(\cdot)$ is the Gamma function. As we are interesting in the motion of a viscoelastic fluid between two coaxial cylinders so the motion will be axially symmetric and the components of velocity can be expressed as $V_r = 0,\ V_\theta = u(r,t)$ and $V_z = 0$. Substituting these velocity components into equation (1) we get

$$\left(1+\lambda^{\alpha} D_t^{\alpha}\right)\tau = \mu\left(1+\lambda^{\beta} D_t^{\beta}\right)\left(\partial_r u - u/r\right) \tag{3}$$

and the balance of linear momentum will leads to

$$\rho \frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tau}{\partial r}\right), \tag{4}$$

where ρ is the constant density of the fluid. Now from equation (3) and (4), we attain the partial differential equation in the form

$$\left(1+\lambda^{\alpha} D_t^{\alpha}\right)\frac{\partial u}{\partial t} = \nu\left(1+\theta^{\beta} D_t^{\beta}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)u(r,t) \tag{5}$$

3. FIRST PROBLEM

Consider coaxial cylinders of radii R_0 and $R_1\ (> R_0)$, and the generalized Oldroyd-B fluid is contained in between two coaxial cylinders. The inner cylinder is kept fixed while the outer cylinder generates the simple harmonic oscillation. So the initial and boundary conditions will be of the form

$$u(r,t) = 0, \quad t = 0 \tag{6}$$

$$u(R_0,t) = 0, \quad t > 0 \tag{7}$$

$$u(R_1,t) = \cos \omega t, \quad t > 0 \tag{8}$$

Introducing the dimensionless quantities

$$u^* = \frac{u}{U_0},\quad r^* = \frac{r}{R_0},\quad t^* = \frac{t}{\nu R_0^2},\quad \eta = \frac{\lambda \nu}{R_0^2},\quad \varepsilon = \frac{\theta \nu}{R_0^2},\quad b = \frac{R_1}{R_0},\quad \omega = \frac{\omega_0 R_0^2}{\nu}. \tag{9}$$

into Eqs. $(12)-(15)$ and dropping the star notation, we get

$$\left(1+\eta^{\alpha} D_t^{\alpha}\right)\frac{\partial u}{\partial t} = \left(1+\varepsilon^{\beta} D_t^{\beta}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)u(r,t) \tag{10}$$

$$u(r,t) = 0, \quad t = 0 \tag{11}$$

$$u(1,t) = 0, \quad t > 0 \tag{12}$$

$$u(b,t) = \cos \omega t, \quad t > 0 \tag{13}$$

Now firstly applying Laplace transform to equations (10), (12) and (13) and then using Webers' transform on the resulting problem, we find

$$\hat{\bar{u}}(\lambda_i, s) = \frac{2}{\pi}\frac{J_1(\lambda_i)}{\lambda_i^2 J_1(\lambda_i b)} \frac{\lambda_i^2 + \lambda_i^2 \varepsilon^{\beta} s^{\beta}}{\left(s + \eta^{\alpha} s^{\alpha+1} + \lambda_i^2 + \lambda_i^2 \varepsilon^{\beta} s^{\beta}\right)} \frac{s}{s^2 + \omega^2} \tag{14}$$

where $J_1(\cdot)$ and $Y_1(\cdot)$ are the Bessel functions of the first and second kind of order one.
Now taking the inverse Webers' transform to equation (14) we get

$$\bar{u}(r,s)=\frac{b\left(r^2-1\right)}{\left(b^2-1\right)r}\frac{s}{s^2+\omega^2}-\sum_{i=1}^{\infty}\frac{\pi J_1\left(\lambda_i b\right)J_1\left(\lambda_i\right)\bar{A}(\lambda_i,s)}{J_1^2\left(\lambda_i\right)-J_1^2\left(\lambda_i b\right)}\phi(\lambda_i,r). \tag{15}$$

in which

$$\bar{A}(\lambda_i,s)=\frac{s+\eta^\alpha s^{\alpha+1}}{\left(s+\eta^\alpha s^{\alpha+1}+\lambda_i^2+\lambda_i^2\varepsilon^\beta s^\beta\right)}\frac{s}{s^2+\omega^2}, \tag{16}$$

and

$$\phi(\lambda_i,r)=J_1\left(\lambda_i r\right)Y_1\left(\lambda_i\right)-J_1\left(\lambda_i\right)Y_1\left(\lambda_i r\right), \tag{17}$$

and λ_i are the positive root of $\phi(\lambda_i,b)=0$.
And now using the next property of the inverse Laplace transform [7, 8] to equation (15)

$$u(r,t)=\frac{b\left(r^2-1\right)}{\left(b^2-1\right)r}\cos\omega t-\sum_{i=0}^{\infty}\frac{\pi J_1\left(\lambda_i b\right)J_1\left(\lambda_i\right)A(\lambda_i,t)}{J_1^2\left(\lambda_i\right)-J_1^2\left(\lambda_i b\right)}\phi(\lambda_i,r). \tag{18}$$

where

$$\begin{aligned}A(\lambda_i,t)=&\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\eta^{\alpha m+\alpha}}\sum_{k=0}^{m}\binom{m}{k}\lambda_i^{2k}\varepsilon^{\beta k}\cos\left(\omega t\right)*t^{\alpha(m+1)+k(1-\beta)-1}\\&\times\left[E_{1+\alpha,\alpha-m+k(1-\beta)}^{(m)}\left(-\lambda_i^2\eta^{-\alpha}t^{\alpha+1}\right)+\frac{\eta^\alpha}{t^\alpha}E_{1+\alpha,k(1-\beta)-m}^{(m)}\left(-\lambda_i^2\eta^{-\alpha}t^{\alpha+1}\right)\right],\end{aligned} \tag{19}$$

and "$*$" represents the convolution of two functions. This equation (18) together with equation (19) gives the velocity field for generalized Oldroyd-B fluid when the outer cylinder generates the simple harmonic oscillation.
Making $\varepsilon\to 0$ in equation (16) we get

$$\bar{A}(\lambda_i,s)=\frac{s+\eta^\alpha s^{\alpha+1}}{\left(s+\eta^\alpha s^{\alpha+1}+\lambda_i^2\right)}\frac{s}{s^2+\omega^2} \tag{20}$$

Applying inverse Laplace transform we have

$$\begin{aligned}A(\lambda_i,t)=&\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cos\left(\omega t\right)*\frac{t^{n\alpha+\alpha-1}}{\eta^{n\alpha+\alpha}}\\&\times\left[E_{1+\alpha,\alpha-n}^{(n)}\left(-\lambda_i^2\eta^{-\alpha}t^{\alpha+1}\right)+\frac{\eta^\alpha}{t^\alpha}E_{1+\alpha,-n}^{(n)}\left(-\lambda_i^2\eta^{-\alpha}t^{1-\beta}\right)\right].\end{aligned} \tag{21}$$

using equation (21) into equation (18) we get the solution for fractional Maxwell fluid [13] when outer cylinder generates the harmonic oscillation.
Again making $\eta\to 0$ into equation (16) and adopting the similar procedure we attain

$$A(\lambda_i,t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\lambda_i^{2n}t^{n+1}E_{1-\beta,\beta n}^{(n)}\left(-\lambda_i^2\varepsilon^\beta t^{1-\beta}\right)*\cos\omega t \tag{22}$$

which together with equation (18) gives the similar solution for fractional second grade.

4. SECOND PROBLEM

For this problem, consider coaxial cylinders of radii R_0 and R_1 $(>R_0)$ and the generalized Oldroyd-B fluid is contained in the annular gap. The inner cylinder is kept fixed while the outer cylinder suddenly rotates with the constant speed U_0. So the governing equation and its corresponding boundary and initial conditions will be the same except the boundary condition (8)

$$u(b,t)=1,\qquad t>0 \tag{23}$$

Adopting the similar procedure, we attain the velocity field in the following form

$$u(r,t)=\frac{b\left(r^{2}-1\right)}{\left(b^{2}-1\right)r}-\sum_{i=0}^{\infty}\frac{\pi J_{1}\left(\lambda_{i}b\right)J_{1}\left(\lambda_{i}\right)A(\lambda_{i},t)}{J_{1}^{2}\left(\lambda_{i}\right)-J_{1}^{2}\left(\lambda_{i}b\right)}\phi(\lambda_{i},r). \tag{24}$$

$$\begin{aligned}A(\lambda_{i},t)=&\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}}{m!\eta^{\alpha m+\alpha}}\sum_{k=0}^{m}\binom{m}{k}\lambda_{i}^{2k}\varepsilon^{\beta k}t^{\alpha(m+1)+k(1-\beta)}\\&\times\left[E_{1+\alpha,\alpha+1-m+k(1-\beta)}^{(m)}\left(-\lambda_{i}^{2}\eta^{-\alpha}t^{\alpha+1}\right)+\frac{\eta^{\alpha}}{t^{\alpha}}E_{1+\alpha,k(1-\beta)-m+1}^{(m)}\left(-\lambda_{i}^{2}\eta^{-\alpha}t^{\alpha+1}\right)\right].\end{aligned} \tag{25}$$

Which gives the behavior of the velocity field when the outer cylinder is rotates with the constant speed. For $\varepsilon \to 0$ one can easily obtain the solution for Maxwell fluid [13], when the outer cylinder rotates with constant speed U_0. And $\eta \to 0$ will give the velocity field for second grade fluid. Making $\eta \to 0$ and $\varepsilon \to 0$ we get the classical solution for Newtonian fluid.

$$u(r,t)=\frac{b\left(r^{2}-1\right)}{\left(b^{2}-1\right)r}-\sum_{i=0}^{\infty}\frac{\pi J_{1}\left(\lambda_{i}b\right)J_{1}\left(\lambda_{i}\right)\exp\left(-\lambda_{i}^{2}t/2\right)}{J_{1}^{2}\left(\lambda_{i}\right)-J_{1}^{2}\left(\lambda_{i}b\right)}\phi(\lambda_{i},r).$$

5. CONCLUSION

The practical importance of the viscoelastic fluid leads to some of challenging equation to the research to obtain the velocity field for such complicated problems. The exact analytic solution for unsteady rotational flow of viscoelastic fluid between coaxial rotating cylinders is obtained by using Laplace and Webers' transforms for two types of flow namely, *(i)* when the outer cylinder makes the simple harmonic oscillation and *(ii)* when the outer cylinder suddenly rotates with the constant speed U_0. It has been shown that the fractional constitutive relationship model is more flexible than the conventional model in describing the properties of viscoelastic fluid. Moreover, the limiting cases of our solution correspond to the similar solutions for fractional Maxwell, fractional second grade and Newtonian fluid performing the same motion.

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