Sindh Univ. Res. Jour. (Sci. Ser.) Vol.45 (1): 113-118 (2013)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

Critical Accelerated Method

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Received 2nd January 2013 and 11th March Revised 2013

Abstract: The numerical solution of linear systems of equations is proposed in this paper. The presented method is a one-parameter reduced generalization of the accelerated overrelaxation (AOR) method. The method is supported with presenting numerical examples in order to check its superiority. Furthermore, the CA method is compared with the AOR method and it is proved an efficient method.

Keywords: Linear systems of equations, CA (or C_{ξ}) method, Overrelaxation.

1. INTRODUCTION

In literature, there are many methods which result in a convergence of systems of linear equations. Some of the methods of linear systems find solutions by direct and the others find it by indirect method. The accelerated overrelaxation (AOR) is one of the direct and indirect methods that play an important role. It is a two-parameter generalization of the successive overrelaxation (SOR) method (Hadjidimos, 1978; Avdelas, 1980 and 1981; Rakhimov, 2009). The SOR is the extrapolation to Gauss-Seidel method that takes the form of a weighted average between the previous iteration and computes the iterations for each component.

In Rakhimov; 2009 the AOR method has been formulated for quarter-sweep iterative scheme applied to the Poisson equation. There are other related methods and the most popular ones are in the literature (Martins, 1980, 1981 and 1983; Niethammer, 1979; Hague, 1987) which played a very important role in solving systems of linear equations. Some of the methods are published in the excellent books by (Varga, 2009 and Young) 2003. The matrix completion problem for a low rank factorization model and a SOR algorithm has been discussed (Wen, 2010) and based on nuclear norm minimization.

Though the Jacobi, Gauss-Seidel and SOR methods perform well but the AOR methods convergence rate is better among them. Many authors have considered sufficient conditions for the convergence of the AOR method (Saridkis, 1986, and 1990; Albrecht, 1984; Missirlis, 1981 and 1984; Papadopoulou, 1989; Dehghan, 2006 and 2007). Dragoslav 1989 has also discussed the convergence criteria of AOR method. The region of the convergence, for the other methods, such as ESOR method, which is presented with the involved parameters, values, in Missirlis, 1984, and Yuan 2003. Accelerating SOR method with another parameter for generalizing SOR method for augmented linear systems under suitable restrictions on the iterations has been described by (Bai, 2005).

The proposed paper that is presented discusses one-parameter reduced generalization of accelerated over-relaxation method. Such a one parameter reduced generalization is called critical accelerated (CA) method.

The well-known method that is presented by Jacobi can be derived as a special case if $\xi = 1$, from the presented method. The method will be called under-acceleration when $0 < \xi \leq 1$ that can be used to obtain convergence of some of the linear system of equations. The method is called over-acceleration when $\xi > 1$ and cannot be used for convergence of the most of the linear systems.

The presented CA method will converge with a condition that the acceleration parameter ξ belongs to $(0, 1]$ if the matrix containing coefficients of the discussed system has some options that it is irreducible diagonally dominant and positive definiteness. For further elaboration some characteristic examples which show the superiority of the CA method are given as a special case.

2. METHOD AND MATERIALS

In this section, the CA method is derived. Let a system of *n* linear equations with *n* unknowns that can be described as follows:

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$$AX = B \tag{2.1}$$

where A has the nonvanishing diagonal elements, real or complex.

The splitting of A matrix can be considered as follows;

$$A \equiv D + A_L^U, \tag{2.2}$$

where

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix},$$

$$A_L^U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & a_{(n-1)n} \\ a_{n1} & \cdots & a_{n(n-1)} & 0 \end{bmatrix}$$

Matrices are the no-vanishing diagonal matrix and the upper-lower coefficient matrix respectively.

For finding the solution of equation (2.1) numerically, the general iterative scheme is being used whose coefficient matrix is of the form

$$\gamma_1 D X^{(k+1)} = (\gamma_2 D + \gamma_3 A_L^U) X^{(k)} + \gamma_4 B, \tag{2.3}$$

where $k = 0, 1, \ldots$ and $\gamma_i \,|\, i = 1(1)4$ with $\gamma_1 \neq 0$, and $X^{(0)}$ be an arbitrary initial guess vector for the solution of equation(2.1).

After dividing $\gamma_1 \neq 0$ on both sides of equation (2.3), the required form is as follows;

$$D X^{(k+1)} = (\gamma_2' D + \gamma_3' A_L^U) X^{(k)} + \gamma_4' B, \tag{2.4}$$

where $\gamma_i' = \gamma_i / \gamma_1 \,|\, i = 2(1)4$ is set.

From the above representation the equations reduce to the following format;

$$\gamma_4' A = (1 - \gamma_2') D - \gamma_3' A_L^U, \tag{2.5}$$

with $\gamma_4' \neq 0$.

For further description, the equation (2.2) follows that;

$$\gamma_2' = 1 - \gamma_4' \text{ and } \gamma_3' = -\gamma_4' \tag{2.6}$$

Following is the parametric solution from the above set of equations where $\gamma_4' = \xi > 0$. Such a parameter is called acceleration parameter and is given below;

$$\gamma_2' = 1 - \xi \text{ and } \gamma_3' = -\xi \tag{2.7}$$

Therefore, (2.4) becomes

$$D X^{(k+1)} = \left[(1-\xi) D - \xi A_L^U \right] X^{(k)} + \xi B, \tag{2.8}$$

or, equivalently,

$$X^{(k+1)} = \left[(1-\xi) I - \xi A' \right] X^{(k)} + \xi C, \tag{2.9}$$

with indeed, the sufficient condition (2.5) becomes satisfactory that is

$$\xi A = \xi (D + A_L^U), \tag{2.10}$$

where, I be the corresponding unity matrix, $A' = D^{-1} A_L^U$ and $C = D^{-1} B$.

Finally, $X^{(k+1)} = X^{(k)}$ is the condition for the solution of system of equation which will be called a convergent solution where the error will be calculated by using the formula $E^{(k)} = X - X^{(k)}$, where X is said to be the exact solution. Thus, the method presented in the equation (2.9) is called Critical Accelerated (CA) method or C_ξ method.

It is observed that for some particular values of the parameter, $\xi = 1$ the CA method reduces to the well-known method of Jacobi, i.e.,

$$X^{(k+1)} = -A' X^{(k)} + C \tag{2.11}$$

Furthermore, the use of the notations is like Λ_ξ for the iterative matrix in (2.9),

$$\Lambda_\xi = (1-\xi) I - \xi A' \tag{2.12}$$

and $\rho(\Lambda_\xi)$ for the spectral radius of Λ_ξ.

It is easily seen that

$$\Lambda_\xi = \xi \Lambda_1 + (1-\xi) I \tag{2.13}$$

It is, therefore, if μ, real or complex, be an eigenvalue of Λ_1 and λ the corresponding one of Λ_ξ, we have

$$\lambda = \xi \mu + (1-\xi) \tag{2.14}$$

Following sections and subsections are for iterations and restrictions that are imposed on the parameter $\xi > 0$ in order to find the convergence of C_ξ method.

Irreducible Diagonally Dominant Matrices

For a matrix A which is an irreducible diagonally dominant matrix that must be non-singular having non-vanishing diagonal elements with $|a_{ii}| > \sum a_{ij}$, $j = 1 \ldots n$, $j \neq i$ the following property which is about the convergence can be stated as follows:

THEOREM-1: For determining $\rho(\Lambda_\xi) < 1$, with the condition that all the eigenvalues of Λ_1 are not greater

than one then an interval $\mathrm{I} = (0,\ \delta)$ can exist for the C_ξ method that converges for all $\xi \in \mathrm{I}$.

Proof: The above stated statement results in that the eigenvalues of $D^{-1}A$ are all-greater than zero. This condition is satisfied because the elements of matrix, which are eigenvalues, should have a property that they are continuous function of their elements and in result it can be said that stated eigenvalues of Λ_1 are also are dis-continuous. This proves that the eigenvalues of Λ_1 are all positive.

Let, $P = \{1,2,3,\ldots p\}$, where P is a natural number, μ_i be an eigenvalue of Λ_1 and λ_i the corresponding one of Λ_ξ for $i \in P$ then we have

$$\lambda_i = 1 + \xi(\mu_i - 1)$$

Since $\mu_i > 0$ and $|\lambda_i| < 1$ requires

$$1 + 2\xi(\mu_i - 1) + \xi^2(\mu_i - 1)^2 < 1$$

$$\text{or} \quad -2\xi(1-\mu_i) + \xi^2(1-\mu_i)^2 < 0$$

$$\text{or} \quad \xi(1-\mu_i) < 2$$

$$\Leftrightarrow \ 0 < \xi < \frac{2}{1-\mu_i}$$

Suppose $\delta = \min_{i \in N}\{2/(1-\mu_i)\}$ and the interval $\mathrm{I} = (0,\ \delta)$ is well defined.

Hence, C_ξ method converges for all $\xi \in \mathrm{I}$.

The following theorem can be stated and proved for complex eignevalues which is as follows:

THEOREM-2: For determining $\rho(\Lambda_\xi) < 1$ for each λ_i that corresponds to Λ_ξ and the eigenvalues of Λ_1 which have real parts are not greater than one then there exists an interval $\mathrm{J} = (0,\ \varepsilon)$ for which C_ξ method converges for all $\xi \in \mathrm{J}$.

Proof: The above stated theorem results in values which are positive from the real parts of eigenvalues of $D^{-1}A$. This is because of the condition that matrix eigenvalues are continuous function so the members, thus the eigenvalues of Λ_1 are not dis-continuous. This proves that there are all positive values for the real parts of eigenvalues of Λ_1. Suppose $S = \{1,2,3,..s\}$, where S is a natural number, μ_i be an eigenvalue of Λ_1 and λ_i the corresponding one of Λ_ξ for $i \in S$, then we have

$$\lambda_i = 1 + \xi(\mu_i - 1)$$

Since $\Re(\mu_i) > 0$ and $|\lambda_i| < 1$ requires

$$1 + 2\xi\Re(\mu_i - 1) + \xi^2|\mu_i - 1|^2 < 1$$

$$\text{or} \quad -2\xi\Re(1-\mu_i) + \xi^2|1-\mu_i|^2 < 0$$

$$\text{or} \quad \xi|1-\mu_i|^2 < 2\Re(1-\mu_i)$$

$$\Leftrightarrow \ 0 < \xi < \frac{2\Re(1-\mu_i)}{|1-\mu_i|^2}$$

Suppose $\varepsilon = \min_{i \in M}\left\{2\Re(1-\mu_i)/|1-\mu_i|^2\right\}$, and the interval $K = (0,\varepsilon)\ \mathrm{J} = (0,\ \varepsilon)$ is well defined.

Thus, C_ξ method converges for all $\xi \in K$ but, if we suppose δ (respectively ε) > 1,

then $1 < \xi < \delta$ (respectively ε) then for this interval the C_ξ method is not convergent. Thus, $\xi \in (0,\ 1]$ the method is established.

3. RESULTS AND DISCUSSION

In this section, the solution of a systems, of equations is discussed. *Maple* is used in the following computation for the well-known method Jacobi as the special case where the hand computational rounded errors can be avoided.

As an example a system of equations which are linear with two equations and two variables is given below;

$$3x_1 - 4x_2 = -1 \text{ and}$$

$$2x_1 - 3x_2 = -1 \qquad (3.1)$$

So, we have

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Therefore, by (2.2) and (2.9)

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}, A_L^U = \begin{bmatrix} 0 & -4 \\ 2 & 0 \end{bmatrix},$$

$$D^{-1} \equiv \begin{bmatrix} 0.333333 & 0 \\ 0 & -0.333333 \end{bmatrix},$$

$$A' = D^{-1}A_L^U \equiv \begin{bmatrix} 0 & -1.333330 \\ -0.666666 & 0 \end{bmatrix} \quad \text{and}$$

$$C = D^{-1}B \equiv \begin{bmatrix} -0.333333 \\ 0.333333 \end{bmatrix}$$

Set $X^{(0)} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\xi = 1$ to iterate using (2.11), we obtain $X^{(3)} = X^{(2)}$ upto 6 decimal points (d.p). Thus,

$$x_1^{(2)} = x_2^{(2)} = 0.999997$$

while the exact solution is:

$$X = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \quad \text{and}$$

$$E^{(2)} \equiv \begin{bmatrix} 0.000003 & 0.000003 \end{bmatrix}^T .$$

For further elaboration of method, a system of equations which are linear with two variables and equations is presented:

$$5x_1 - 2x_2 = 1 \text{ and } 2x_1 - x_2 = 0 \quad (3.2)$$

So, we have

$$A = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, by (2.2) and (2.9)

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_L^U = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},$$

$$D^{-1} \equiv \begin{bmatrix} 0.200 & 0 \\ 0 & -1.000 \end{bmatrix},$$

$$A' = D^{-1}A_L^U \equiv \begin{bmatrix} 0 & -0.400 \\ -2.000 & 0 \end{bmatrix} \quad \text{and}$$

$$C = D^{-1}B \equiv \begin{bmatrix} 0.200 \\ 0 \end{bmatrix}$$

Set $X^{(0)} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $\xi = 1$ after a number of iterations using (2.11), the result can be obtained which is found after $X^{(42)} = X^{(41)}$ and is correct to three decimal places. Thus,

$$x_1^{(41)} = 0.992,\ x_2^{(41)} = 1.980$$

But if the approximate values are compared with exact solutions then the results are as follows,

$$X = \begin{bmatrix} 1 & 2 \end{bmatrix}^T \text{ and } E^{(41)} \equiv \begin{bmatrix} 0.008 & 0.020 \end{bmatrix}^T .$$

3.1 Comparison of CA method with AOR method

The presented method performs well on a system of linear equations discussed earlier in section 3. The numerical example discussed in section 3, i.e.,

$$3x_1 - 4x_2 = -1 \quad \text{and}$$
$$2x_1 - 3x_2 = -1$$

has also been discussed in Hadjidimos, 1978 using AOR method. The approximate solution has been calculated in 27th iteration using AOR method whereas the CA method calculates the same in the 3rd iteration. Thus, the CA method shows its efficiency over the AOR method.

4. CONCLUSION

The paper described a CA method that is a simple and a powerful technique for solving linear system of equations. The presented method is an efficient technique, which can find the approximate solutions of certain partial differential equations. The presented method performs well when compared with the well-known methods, such as Gauss -Seidel, Accelerated Over-relaxation and Successive over-relaxation methods.

The CA method converges faster for linear system of equations, when $0 < \xi < 1$.

The proposed method can be extended in future for determining the intervals suitable for including optimum ξ with a condition that the fast convergence is obtained.

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