



A Posteriori Control of Regularization for Complementary Image Motion Problem

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Abstract: An adaptive control of regularization is presented in this paper which is based on a posteriori error estimation for a variational optic flow model which is designed using complementary approach. In this paper the adaptive control and regularization technique is improved and extended as appeared in the paper zakaria (2011). It is shown that with the improvement in the data term, the successful and fast “a posteriori” error control is obtained with a significant improvement in regularization process and determination of dense optic flow field. This method is based on the adaptive finite element method using unstructured grid as the discrete computational domain which allows the locally adaptive choice of optimal regularization parameters. The given Meshes on unstructured grid and the dramatic improvement in flow field at various adaptive iterations is the core of this presentation and could be an attraction for the image community.

Keywords: Optic Flow, regularization, Adaptive Control of regularization, Adaptive finite elements, optimization.

1. INTRODUCTION

The estimation of the two dimensional disparity maps from subsequent images of a video sequence is one of the classical image analysis problems and the active research area in the computer vision research. This velocity image is called optic flow, in other words the optic flow is a dense vector field which associates one displacement vector to each pixel in the image plane. Reliable computation of this displacement field constitutes one challenging task in the computer vision research. Image research community is very much attracted to this research area of image sequence analysis; the reader is referred to zakaria (2011), Amur (2011), Amur (2012), Alvarez (1999), Anandan (2000), Aubert (1995), Brozi (2002), Bruhn (2002), Bruhn (2005), (Bruhn (2006) Horn-Shunk(1981), Issasque (1993), kumar (1996), Lucas(1981), Nagel (1983), Nagel (1986), Nagel(1990), Schnör (1994), Weickert (2005), Weickert (2005), Zimmer (2009) and references therein. The variational approach is one of the modern and most efficient approaches proposed to overcome the ill-posed problems in image sequence analysis Zakria (2011), Amur (2011), Amur (2012), Bruhn (2005), Weickert (2005), Bruhn (2002), Aubert (1995).

Considering the variational approach one calculates the flow field as a minimizer of a suitable energy functional. The variational approaches are successful in vision problems because the variational model consists of a data term based on optic flow constraints (constancy assumptions), e.g grey value

constancy in Horn and Schunck model Horn (1981) and a regularization term is used to control the oscillations in the disparity field consequently a fully dense optic flow is recovered. However there is an ambiguity that these two terms may contradict each other, to overcome this difficulty a novel approach was proposed as complementary optical flow Zimmer, (2009) where they have focused on improvements in data term (constancy assumptions) and smoothness term to create complementarity between these two terms. The variational method in this paper is extension of the results appeared in Belhachmi (2011, by improving the data term in their variational model using complementarity approach in Zimmer, (2009). The Variational methods are ambiguous in reliability of computations at all locations of the given image geometry, which is their major drawback. Our goal is to show that the novel approach of the control of the regularization effects as proposed in Belhachmi (2011), Amur (2011), Amur (2012) is a powerful tool for problems such as the estimation of optic flow. Here an adaptive finite element method and triangular grid is used as a computational domain to compute the optic flow field. The adaptation is performed using a posteriori error indicators as discretization error and serve as novel regularization devices. The scaling factors in the error indicators are considered as local smoothing parameters. To show the robustness of adaptation procedure some simulations in the form of

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“mesh” are given for the optic flow at every iteration of the given adaptive algorithm.

Variational Optic Flow Problem: The vector field $u = (u_1, u_2)$ describing the motion in a given image sequence of an image is called optical flow. The sequence of an image in the domain $\Omega \in \mathbb{R}^2$ is modeled by $u : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$. The estimation of optical flow is based on the image brightness constancy:

$$f(x_1, x_2, t) = f(x_1 + \partial x_1, x_1 + \partial x_2, t + \partial t) \dots (1)$$

First order Taylor expansion yields:

$$\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial f}{\partial t} = 0 \dots \dots \dots (2)$$

For $f \in C^1([0, \infty); \mathbb{R})$, The problem is to find the displacement field $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ such that:

$$f_t + f_{x_1} u_1 + f_{x_2} u_2 = 0 \dots \dots \dots (3)$$

Where the vector field $u = (u_1, u_2)$ is called the unknown velocity field called optic flow. The variational approach leads to determine $u = (u_1, u_2)$ as the minimizer of a given energy model. The two parts of the given functional consist of the data term is derived from constancy assumption (3) and a regularization term to overcome the ill-posed problem (One equation and two unknowns (3)). For overview of optic flow models the reader is referred to [Bruhn (2002), Bruhn (2005)] and the references given in this paper. Using the variational approaches as Belhachmi (2011), Bruhn(2005) the complementary model is proposed as given in Zimmer(2009). The following energy functional is considered which is associated to the optic flow recovery by variational methods:

$$E(u) = \int_{\Omega} (u^T \frac{1}{|\nabla f|^2 + \zeta^2} J_{\rho} u + \alpha(x) |\nabla u|^2) dx + \int_{\Omega} (\frac{1}{|\nabla f|^2 + \zeta^2} f \cdot u) \dots \dots \dots (4)$$

The $\alpha(x)$ is strictly positive scalar valued continuous smoothing function on domain Ω . Here $x = (x_1, x_2)^T$ accordingly, a disjoint partition is considered as a finite number of sub domains Ω_i such that the function

$\alpha(x)$ is equal to α_i on each Ω_i . Assume that $\alpha_{\min} > 0$.

The variational optimization problem: find $u(\Omega)$ such that

$$E(u) = \inf E(v)$$

The structure tensor J_{ρ} is the matrix given as:

$$J_{\rho} = \begin{pmatrix} K_{\rho} * f_{x_1}^2 & K_{\rho} * f_{x_1} f_{x_2} \\ K_{\rho} * f_{x_1} f_{x_2} & K_{\rho} * f_{x_2}^2 \end{pmatrix}$$

And $K_{\rho}(x, y)$ is a Gaussian kernel with standard deviation ρ , $*$ stands for convolution. For sake of simplicity the data normalization parameter θ is introduced as

$$\theta = \frac{1}{|\nabla f|^2 + \zeta^2}$$

Where the regularization parameter $\zeta > 0$ is used to avoid the division by zero. For a vector field w the gradient is denoted as

$$grad(w) = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} \end{pmatrix}$$

Set $f = (-\theta f_{x_1} f_t, -\theta f_{x_2} f_t)^T$, and $u = (u_1, u_2)^T$, where f_{x_1} , f_{x_2} and f_t are denoted as the derivatives with respect to x_1 , x_2 and t .

The work is done in H^1 functional space, for other possible choices the reader is referred to [Belhachmi(2011), Amur(2011)]. It is assumed that the functional space H^1 as a Hilbert space for the scalar product

$$(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} \theta J_{\rho} u \cdot v dx \text{ with}$$

the norm

$$\|u\|_{H^1(\Omega)} = (u, u)^{\frac{1}{2}}$$

The Euler-Lagrange equations are given by:

$$\begin{cases} \operatorname{div}(\alpha(x)\nabla u_1) - (\theta f_{x_1}^2 u_1 + \theta f_{x_1} f_{x_2} u_2 + \theta f_{x_1} f_t) = 0 \\ \operatorname{div}(\alpha(x)\nabla u_2) - (\theta f_{x_1}^2 u_1 + \theta f_{x_1} f_{x_2} u_2 + \theta f_{x_2} f_t) = 0 \end{cases} \quad (5)$$

$$\text{where } \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0, \text{ on } \partial\Omega.$$

Remark 2.3 In the usual optic flow models [Bruhn (2005)] the $\alpha(x)$ is taken constant in the whole domain, but here is taken as function of space variable. It is assumed that it is known a priori.

The weak formulation of (5) reads:

$$\begin{cases} \text{Compute } u \in H^1 \text{ such that} \\ \int_{\Omega} (\alpha(x)\nabla u : \nabla v) dx + \lambda \int_{\Omega} \theta J_{\rho} u \cdot v dx \dots\dots (6) \\ = \int_{\Omega} \theta f \cdot v dx \quad \forall v \in H^1 \end{cases}$$

Where for $u, v \in H^1(\Omega)$

$$\begin{aligned} b_{\alpha}(u, v) &= \int_{\Omega} \alpha(x) \nabla u \cdot \nabla v dx + \lambda \int_{\Omega} \theta J_{\rho} u \cdot v dx \\ (f, v) &= \int_{\Omega} \theta f \cdot v. \end{aligned}$$

Therefore from (6), compute $u \in H^1(\Omega)$ such that:

$$b_{\alpha}(u, v) = (f, v), \quad \forall v \in H^1(\Omega)$$

For a posteriori error control the constants are considered as independent of α , for estimation of a posteriori error estimation following norm is introduced as:

$$\|u\|_{\rho, \alpha}^2 = \left\| \alpha^{\frac{1}{2}} \nabla u \right\|_{L^2(\Omega)}^2 + \left\| \theta^{\frac{1}{2}} J_{\rho}^{\frac{1}{2}} u \right\|_{L^2(\Omega)}^2$$

For the a posteriori analysis purpose, it is assumed that the matrix J_{ρ} is definite a.e.

Problem Discretization:

The discrete domain is considered as is triangular grid T_h made up of elements with maximum size $h > 0$. Following discrete space as $X_h \subset X$, is introduced which denotes the space of continuous piecewise affine finite elements over T_h , that is

$$X_h := \left\{ v_h \in C^0(\bar{\Omega}) \mid \forall K \in T_h, v_h|_K \in P_1(K) \right\}$$

Where $P_1(K)$ denotes space of all polynomials in R^2 of degree equal to one.

For any $K \in T_h$ the discrete version of structure tensor is denoted as $J_{\rho, h, K}$ where

$$(\theta J_{\rho, h, K})_{i, j} = \frac{1}{|K|} \int_K (\theta J_{\rho})_{i, j} dx, \quad 1 \leq i, j \leq 2$$

Problem Discretization is given as

$$\begin{cases} \text{Compute } u_h \in X_h, \text{ where} \\ b_{\alpha_h}(u_h, v_h) = (f, v_h) \quad \forall v_h \in X_h \dots\dots (7) \end{cases}$$

Where

$$b(u_h, v_h) = \int_{\Omega} \alpha(x) \nabla u_h \cdot \nabla v_h dx + \lambda \int_{\Omega} u_h \cdot v_h dx$$

and

$$(f, v_h) = \int_{\Omega} f \cdot v_h.$$

To assure the ellipticity of the bilinear form J_{ρ} is assumed as positive definite matrix. From the Lax-Milgram Lemma:

Proposition: For any $\alpha > 0$ there exists a unique solution of the discrete problem (7)

A posteriori Error Indicator:

The residual error estimator for the problem (7) is defined by the formula as given in Verfurth (1996), zakaria (2011), Amur (2011).

For each triangle (element) $K \in T_h$,

$$\begin{aligned} \eta_K &= \alpha_K^{\frac{1}{2}} h_K^2 \left\| \theta f + \alpha_K \Delta u_h + \theta J_{\rho, h} u_h \right\|_{L^2(K)}^2 \\ &+ \frac{1}{2} \sum \alpha_e^{\frac{1}{2}} h_e^{\frac{1}{2}} \left\| [\alpha_e \nabla u_h \cdot n_e]_e \right\|_{L^2(e)}^2 \dots\dots (8) \end{aligned}$$

Where $\alpha_e = \max(\alpha_{K_1}, \alpha_{K_2})$, is the maximum value on K_1 and K_2 are two triangles corresponding to the edge e . $[\varphi]_e$ the jump of the φ across the edge e in the direction of normal vector n_e . h_K is denoted as the diameter of K and h_e is the length of $e \in \mathcal{E}_K$ (set of edges).

2. MATERIALS AND METHODS

A posteriori control and simulation results

The numerical experiments are performed using FreeFem++. For further details the reader is referred to the FreeFem++ as a programming language

for numerical solution of partial differential equations by F.Hecht (<http://www.freefem.org/>).

Adaptive Algorithm:

A novel strategy is used in this paper for the adaptive procedure which is different from usual refinement techniques. This is an interesting method which is based on coarsening the initial grid where it is required. Especially in this case when it is observed from the experiments that the refinement of grid is performed by increase in grid points on discontinuities like corners and other damaged regions where the error indicator density is large, consequently the coarsening of grid appears in the regions away from corners. This whole procedure is performed automatically showing the dramatic regularization effects on optic flow grid. The algorithm (Zakaria (2011), Amur (2011), Amur (2012)) is based on the following steps:

(1) Start the initial triangular grid T_h^0 associated to the given image.

(ii) Adaptive procedure

- Compute the flow $u_{\alpha_0, h}$ on T_h^0 grid to the noisy image with $\alpha = \alpha_0$ large constant.
- Build the isotropic triangular mesh T_h^1 using the metric as residual error indicator (8) which is well suited for the two dimensional disparity $u_{\alpha_0, h}$.
- Execute the local optimal choice of $\alpha(x)$ on T_h^1 is and set a new function $\alpha_1(x)$.

(iii) GO to step (ii) and compute $u_{\alpha_1, h}$ on T_h^1

For more detailed information about this algorithm the reader is referred to the references Zakaria(2011) (Amur2011, Amur(2012)).

- For automatic adaptive choice of optimal regularization parameter α , during adaptation process for each triangle K , following formula is proposed
-

$$\alpha_K^{k+1} = \max \left(\frac{\alpha_K^k}{1 + \beta^* \left(\left(\frac{\eta_K}{\|\eta\|_\infty} \right) - 0.1 \right)^+}, \gamma_\alpha \right) \dots \textcircled{9}$$

where γ_α is a thresholding parameter and β is a used for controlling the variations in α in damaged regions of optic flow field. The principal of the adaptive control of regularization process is to choose the damaged

regions where the a posteriori error density η_K is increasing and decrease the values of α to refine the grid in those damaged regions. The optimal choice of regularization parameter α is considered as constant in the regions of the flow field where the solution is already smooth enough. Here $(u)^+ = \max(u, 0)$ and $\|\eta\|_\infty = \max(\eta_K)$. This process of regularization decreases the error density η_K from approximate solution. The results of convergence are shown in the given table 1, for three adaptive iterations and the plot Figure1. The algorithm is intelligent in the sense that it work on the rule of regularization where required. The regularized optic flow is obtained with a less number of triangles (elements) (computational efficiency) and a map (the graph of $\alpha(x)$).

3. RESULTS AND DISCUSSION

Example Taxi Image Sequence

The taxi image sequence is considered for experiment which is downloaded from <http://i21www.ira.uk.de/>. First two frames of taxi image sequence (256 x 191 gray value images) are considered here to compute the optic flow using complementary optic flow model. Four adaptive iterations of the given adaptive algorithm are performed to show the adaptive control for given variational model. The process of adaptation converges up to first three iterations. The results showing a posteriori control are shown in (Table-1) and plot (Fig.1). The numerical results (as images) are given here in Figure 2 as the vector plots, the plots showing the magnitude of optic flow, adapted meshes obtained from various adaptive iterations, error indicators η_K , and the plots for distribution of the $\log_{10} \alpha$ which are shown on unstructured grid as the computational domain. The robustness of adaptive algorithm is quite clear from the given numerical simulations. Moreover the performance of the algorithm is more or less improved as compared to the results given in the paper Belhachmi (2011), specifically when the adaptive procedure is observed from the meshes on various iterations, the speed of adaptation is increased. Dramatic refinement and coarsening of grid improves the quality of solution at each adaptive level. One can observe from the plots for error density η_K that the error density which is initially large at edges and decreases at every adaptive iteration producing an improved dense flow field. From mesh adaption process it is observed that the number of elements increases beside the edges while the coarsening of the mesh occurs outlying from these locations. Initially the large value for α_h is taken. For

this example the initial value is selected as $\alpha_h = 1000$

The interesting plots for $\log_{10} \alpha$ assure the successful and an improved implementation of given adaptive algorithm. One can compare the plots of $\log_{10} \alpha$ after first iteration and third iteration that the value of α_h decreases on image edges (regions with large error density) and increases or remains stable away from these regions. The some color images are given as distributions of $\log_{10} \alpha$ and η_K , where the regions with darker colors show the greater values and lighter colors show the smaller values. The numerical results for this example are given in (Fig. 2), though the complementary optic flow model has improved the performance of the algorithm, which successfully separated the moving object in the image sequence from static objects (optic flow). but one can observe from the plots of flow field and the mesh on iter 3 (Third iteration) that the smooth edges are obtained which is because of the extra smoothing properties of quadratic regularizer, to cope with this difficulty the non quadratic regularization is proposed that will appear in our forthcoming papers.

Table .1 Residual Error Indicator

Iteration No.	Error Indicator
1	0.0260595
2	0.0060970
3	0.004609

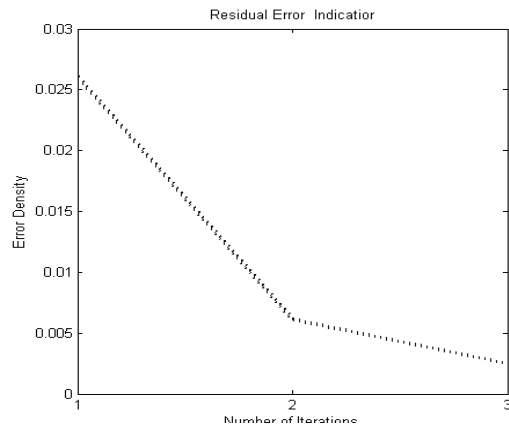
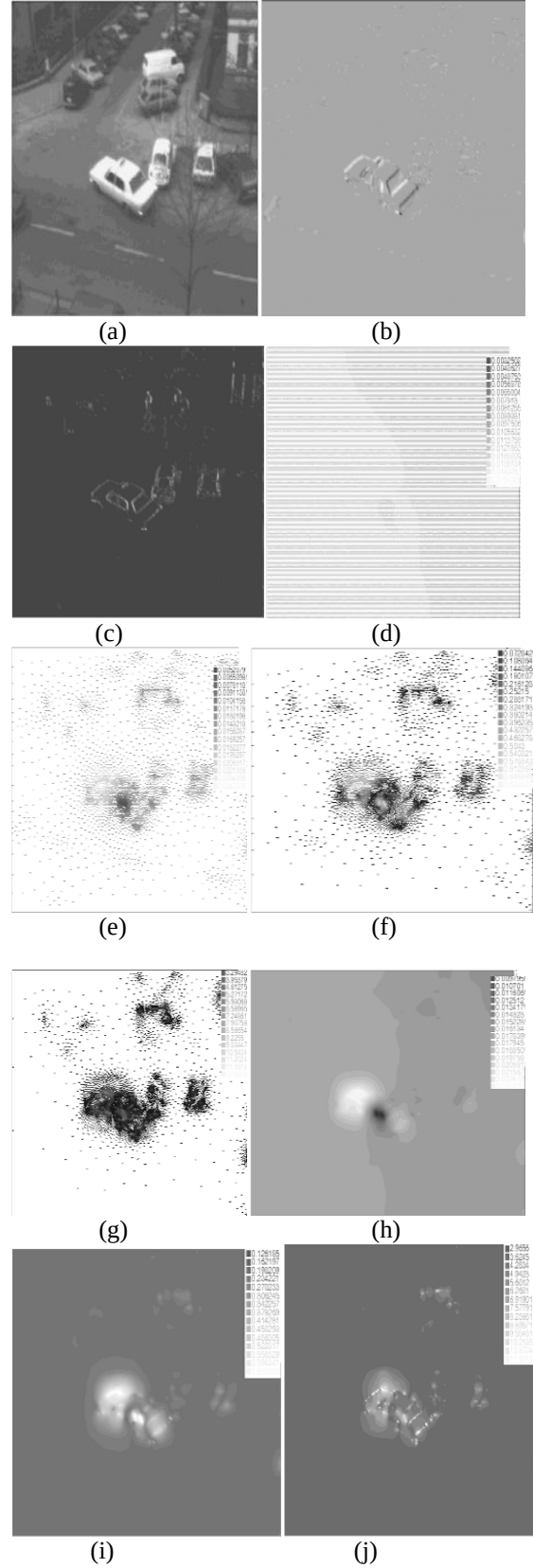


Fig. 1: . Error Indicator Density at Various Adaptive iterations.



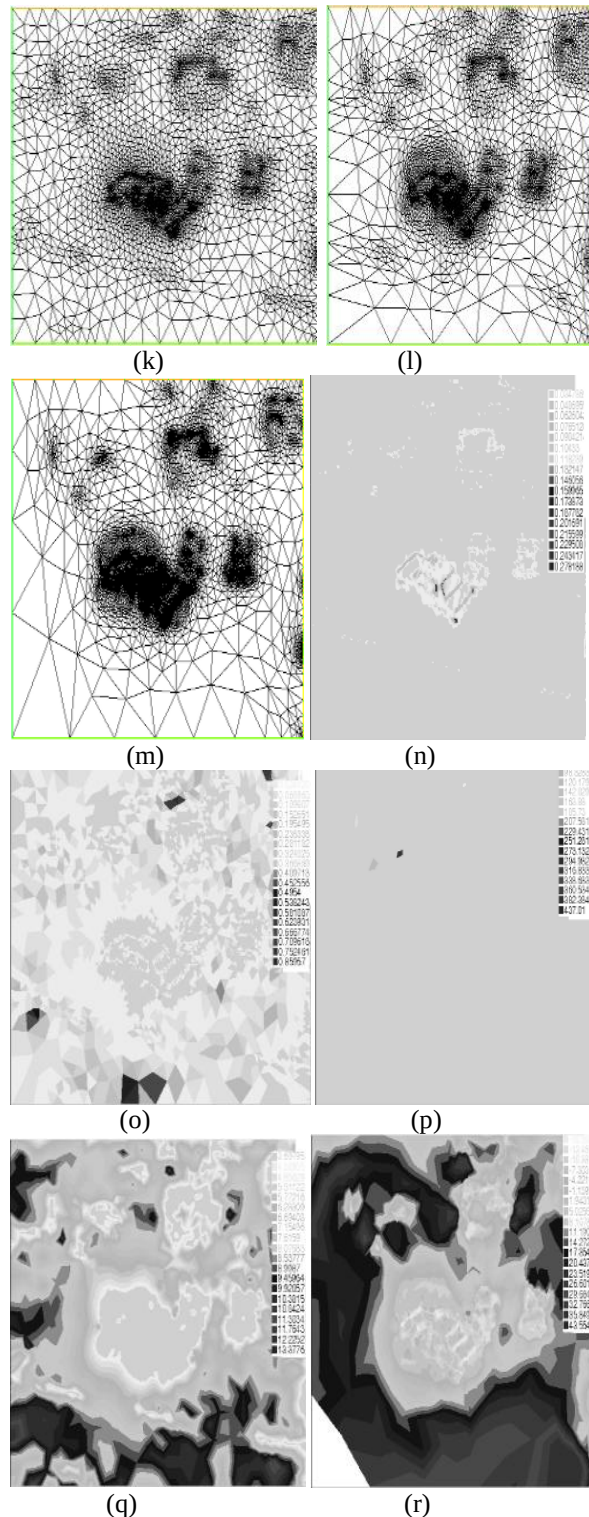


Fig. 2: (From left to right and from top to bottom) (a) First image frame of Humburg Taxi sequence, (b) time derivative $(f_1 - f_2)$. (c) structure tensor J_ρ , (d) Optic flow vector at initial grid, (e) Optic flow vector after first adaptive iteration (f) Optic flow vector after second adaptive iteration (g) Optic flow vector after

third adaptive iteration (h) Optic flow magnitude after first iteration (i) Optic flow magnitude after second iteration (j) Optic flow magnitude after third iteration (k) mesh after first adaptive iteration (l) mesh after second adaptive iteration (m) mesh after third adaptive iteration (n) Error indicator density η_K after first adaptive iteration (o) Error indicator density η_K after third adaptive iteration (p) Error indicator density η_K after third adaptive iteration (q) Effects of regularization by α_h after first adaptive iteration (r) Effects of regularization by α_h after third adaptive iteration.

4.

CONCLUSION

A novel adaptive strategy Belhachmi (2011), Amur (2011), Amur (2012) is successfully implemented on complementary optic flow model (4) based on quadratic regularization. Dense optic flow field was observed. More over with the modification in mathematical model, the dense optic flow with very fast convergence process is achieved. Because of extra regularization effects by quadratic regularizer the main interest goes to apply this strategy in forth coming paper to the nonlinear models based on TV regularization which is discontinuity preserving approach.

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