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A Newiterative Integrator for Cauchy Problems

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Abstract: The paper proposes a new iterative integrator capable of solving first order initial value problems (IVPs), also called Cauchy problems, in ordinary differential equations (ODEs). The integrator was found to be superior to Forward Euler's Method (FoEM) and achieves second order accuracy making it comparable to Ralston's, explicit trapezoidal and midpoint methods. Consistency, Convergence and Stability are met. The derivation of the scheme depends on the idea of combining a constant and an exponential function. The integrator is numerically tested and compared with methods discussed in literature. Numerical computations have been carried out via MATLAB programming language.

Keywords: Initial Value Problems (IVPs), Cauchy problems, Forward Euler's Method, Iterative Integrator.

1. INTRODUCTION

The field of differential equations (ordinary and partial) is a versatile domain to work in. The field is vibrating for mathematical modeling and studying physically and naturally occurring phenomena. Its applicability involves variety of topics from applied and computational mathematics to biotechnologies and business industries. Most of such mathematical models do not have closed form solutions causing one to explore best possible approximate solution as reported in (Burden, *et al*., 2010).

The theory of linear differential equations is well – defined and comprehensible. On the other hand, the philosophy of non – linear differential equations is somehow complicated and requires deep insight (Edwards, *et al*., 2008).

Advent of digital computers energized many researchers interested in the field since this invention reduced considerable time and effort in order to get numerical solutions to differential equations. Currently, various programming languages can execute results generating from numerical algorithms with fastest possible speed and accuracy. The most popular programming language, among scholars, is MATLAB because of its available numerical routines and graphics tools.

In this paper we have developed a new iterative integrator capable of solving Cauchy problems of the form,

$$\frac{dy(t)}{dt} = f\left(t, y(t)\right) \; ; \; y(t_0) = y_0 \qquad (1.1)$$

where $a \le t_0 \le b$ and a, b are finite real numbers.

As stated in (Palais, *et al.*, 2009) that there is no unique method to solve every type of differential equation. Many researchers and scholars have derived numerical schemes when ODEs are divided into classes and further subclasses (Jhanke, 2003). Studies show that most of these schemes have been developed by representing actual solution $y(t)$ to equation (1.1) by an interpolating (linear or nonlinear) function of the form $I(t) = a_0 + a_1 t + b\left[\text{real}\left(e^{(\rho t+\mu)}\right)\right]$ in the interval $\left[t_k, t_{k+1}\right]$, where a_0, a_1 and b are real undetermined coefficients, whereas ρ and μ are complex parameters. For more details, see (Fatunla, 1976), (Hellemans, 1982), (Ibijola, 1997) and (Sunday, *et al.*, 2012).

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Curiosity compels many researchers to explore new iterative integrators and the story starts with the birth of a renowned English Mathematician Brook Taylor. Despite its efficiency and accuracy, there came few instances where Taylor Series Method was beaten by other algorithms (Chandio *et al*., 2010). One of the most governing disadvantages of the method was computing successive derivatives. Picard's method, on the other hand, gave some satisfactory results, though it is also practically a non-recommended method because of complicated successive approximations of integrals as suggested by (Palais, *et al*., 2009).

Derived from Taylor Series Method, first order Forward Euler's Method became foundation of many efficient iterative integrators. Mathematically,

$$y_{k+1} = y_k + \Delta t\, f\left(t_k, y_k\right) + O\left(\Delta t^2\right) \qquad (1.2)$$

where $\Delta t\ \left(= t_{k+1} - t_k\right)$ is the time step length and $t_{k+1} = t_k + \left(k+1\right)\Delta t\,;\ k = 0, 1, 2, \cdots$

Besides being conditionally stable, the method requires extremely small step length to better approximate the solution to (1.1) thereby making it non – recommended for practical purposes as reported in (Kreyszig, *et al*., 2011).

Runge (1901) proposed second order methods giving better accuracy than Euler and, in addition to this, were less cost effective with respect to computer usage. Similarly, (Chandio, *et al*., 2010) proposed a new scheme to increase efficiency of Heun's Method.

A computationally reliable numerical integrator, based on the method introduced by (Fatunla, 1976), was presented by (Sunday, *et al.*, 2012) using combination of polynomial, exponential and cyclometric functions. But the integrator lacked discussion of stability and order of the method introduced. On the other hand, a non – standard explicit algorithm developed by (Ramos, 2007) was found to be *second order accurate* and *stable*.

Others include those developed by (Ademiluyi, 1987), (Ibijola, 1997), (Kama, *et al*., 2000), (Wazwaz, 2000), (Li, 2007), (Ibijola, *et al*., 2010), and (Ibijola, *et al*., 2011), to mention a few.

2. MATERIALS AND METHOD

The derivation of the proposed method is based upon the work of many researchers. The authors have come up with a new iterative integrator which will later be used to modify one of the Euler's variants. The new numerical integrator consists of a constant function and an exponential function of the type

$$I\left(t\right) = A_0 + A_1 e^{\lambda t} \qquad (2.1)$$

where A_0, A_1 and λ are real undetermined coefficients. Let y_k and y_{k+1} being numerical approximants to theoretical solutions $y\left(t_k\right)$ and $y\left(t_{k+1}\right)$ respectively can be expressed as $I\left(t_k\right) = y_k$ and $I\left(t_{k+1}\right) = y_{k+1}$ and $f_k = f\left(t_k, y_k\right)$. Considering (2.1), we proceed in the following way:

$$I\left(t_k\right) = A_0 + A_1 e^{\lambda t_k} = y_k \qquad (2.2)$$

$$I\left(t_{k+1}\right) = A_0 + A_1 e^{\lambda t_{k+1}} = y_{k+1} \qquad (2.3)$$

Differentiate (2.2) to get

$$I'\left(t_k\right) = A_1 \lambda e^{\lambda t_k} = f_k \qquad (2.4)$$

Subtracting (2.2) from (2.3) and using (2.4), we get

$$y_{k+1} = y_k + \frac{f_k}{\lambda}\left(e^{\lambda \Delta t} - 1\right);\ \ 0 < \lambda \le 1$$

For the sake of convenience; λ=0.5 (say). Then our new interpolant becomes

$$y_{k+1} = y_k + 2\left(e^{0.5\Delta t} - 1\right) f_k \qquad (2.5)$$

Averaging the slope at the initial point (given) and the slope at one step marched forward (new interpolant), we obtain

$$K = 0.5\left[\, f\left(t_k, y_k\right) + f\left(t_{k+1}, y_{k+1}\right)\right]$$

Finally; using one of the Euler's variants (midpoint rule), we get

$$y_{k+1} = y_k + \Delta t\, f\left(t_k + \Delta t/2, y_k + \Delta t/2 * K\right)$$

Writing everything at one place, we obtain single step explicit method shown below:

$$y_{k+1} = y_k + \Delta t\ f\left(\begin{array}{l} t_k + \dfrac{\Delta t}{2}, y_k + \dfrac{\Delta t}{4} * \\ (f(t_k, y_k) + f(t_k + 2\Delta t(e^{0.5\Delta t} - 1), \\ y_k + 2\Delta t(e^{0.5\Delta t} - 1) f_k)) \end{array}\right) \quad (2.6)$$

The proposed iterative integrator (2.6) is found to be of second order with local truncation error $O(\Delta t^3)$ and produced better results than many second order methods of literature when compared.

3. RESULTS AND DISCUSSION

The developed integrator is now implemented on two Cauchy problems of the type (1.1) chosen from the literature and computations are carried out using MATLAB. As far as first problem is concerned, results obtained are compared with that of Forward Euler's, Explicit Trapezoidal and Ralston's Methods with respect to absolute percentile errors and the same are also shown by error plot; whereas graphical comparison with exact solution is made for the second problem. Besides, maximum absolute errors have also been computed for problem 2 in table 2 using some selected number of integration steps (NS). Maximum of absolute errors is defined as

$$e_{\max} = \max_{t_k \in [a,b]} \left[\left| y(t_k) - y_k \right| \right].$$

Being frequently used in Statistics, Medicine, Neural Networks and Chemistry; following Logistic Model has been selected from (Sunday *et al*., 2012) to test the performance of newly proposed integrator:

Problem 1: $\dot{y} = y(1-y), y(0) = 0.5$

Analytical solution: $y(t) = 1/(1+e^{-t})$

Table 1. Absolute Percentile Errors of $y(t)$ in problem 1, using proposed integrator and other methods for the finite interval [0,10] with Δt=0.005, 0.025, 0.05 and 0.1. (time t is given non-dimensional)

Δt	at time *(t)*units	Forward Euler's Method	Explicit Trapezoidal Rule	Ralston's Method	Proposed Integrator
0.005	1	1.6147e-02	3.4438e-05	1.4986e-05	4.5308e-06
	2	2.5875e-02	4.2703e-05	2.8478e-05	1.4074e-05
	3	2.0294e-02	3.2549e-05	2.5819e-05	1.8912e-05
	4	1.1911e-02	1.9202e-05	1.6483e-05	1.3646e-05
	5	6.0596e-03	9.8494e-06	8.8138e-06	7.7145e-06
	6	2.8473e-03	4.6598e-06	4.2739e-06	3.8569e-06
	7	1.2740e-03	2.0962e-06	1.9536e-06	1.7966e-06
	8	5.5169e-04	9.1160e-07	8.5904e-07	8.0015e-07
	9	2.3339e-04	3.8701e-07	3.6766e-07	3.4559e-07
	10	9.7022e-05	1.6137e-07	1.5425e-07	1.4599e-07
0.025	1	8.0625e-02	8.6655e-04	3.7685e-04	1.2078e-04
	2	1.2979e-01	1.0784e-03	7.1869e-04	3.3645e-04
	3	1.0166e-01	8.2351e-04	6.5294e-04	4.5986e-04
	4	5.9447e-02	4.8630e-04	4.1732e-04	3.3342e-04
	5	3.0109e-02	2.4959e-04	2.2331e-04	1.8890e-04
	6	1.4081e-02	1.1813e-04	1.0834e-04	9.4557e-05
	7	6.2699-03	5.3156e-05	4.9534e-05	4.4080e-05
	8	2.7019e-03	2.3122e-05	2.1787e-05	1.9643e-05
	9	1.1374e-03	9.8176e-06	9.3262e-06	8.4872e-06
	10	4.7051e-04	4.0941e-06	3.9133e-06	3.5863e-06
0.05	1	1.6096e-01	3.4939e-03	1.5182e-03	5.2087e-04
	2	2.6066e-01	4.3686e-03	2.9088e-03	1.2665e-03
	3	2.0380e-01	3.3440e-03	2.6497e-03	1.7717e-03
	4	1.1862e-01	1.9774e-03	1.6962e-03	1.2927e-03
	5	5.9739e-02	1.0158e-03	9.0855e-04	7.3457e-04
	6	2.7770e-02	4.8105e-04	4.4107e-04	3.6832e-04
	7	1.2290e-02	2.1656e-04	2.0177e-04	1.7189e-04
	8	5.2636e-03	9.4232e-05	8.8780e-05	7.6654e-05
	9	2.2022e-03	4.0024e-05	3.8016e-05	3.3141e-05
	10	9.0544e-04	1.6695e-05	1.5956e-05	1.4010e-05
0.1	1	3.2069e-01	1.4195e-02	6.1554e-03	2.3889e-03
	2	5.2569e-01	1.7923e-02	1.1911e-02	4.3979e-03
	4	2.3608e-01	8.1799e-03	7.0101e-03	4.8114e-03
	6	5.3956e-02	1.9973e-03	1.8303e-03	1.3851e-03
	8	9.9761e-03	3.9217e-04	3.6934e-04	2.8956e-04
	10	1.6746e-03	6.9616e-05	6.6511e-05	5.3070e-05

In order to visualize the behavior of errors calculated in table 1; the error plot has been drawn on logarithmic scales for one of the step sizes Δt=0.005.

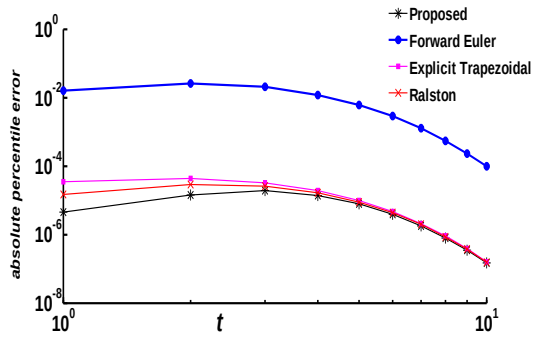


Figure 1: Error comparison for problem 1 with step size Δt=0.005

Problem 2: $\dot{y} = 4t - 2ty,\ y(0) = 3$ (3.2)

Analytical solution: $y(t) = 2 + e^{-t^2}$

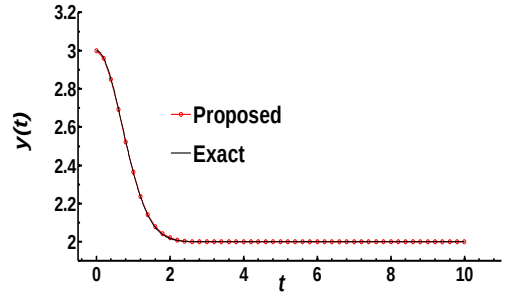


Table 2. Maximum absolute errors for problem 2.

NS	FoEM	Trapezoidal	Ralston	Proposed
50	7.51237e-02	7.08962e+04	3.53483e+04	4.06276e-03
100	3.48031e-02	2.05217e-03	1.40559e-03	1.02033e-03
200	1.68586e-02	4.79721e-04	3.23437e-04	2.54679e-04
400	8.28385e-03	1.16206e-04	7.77865e-05	6.34858e-05

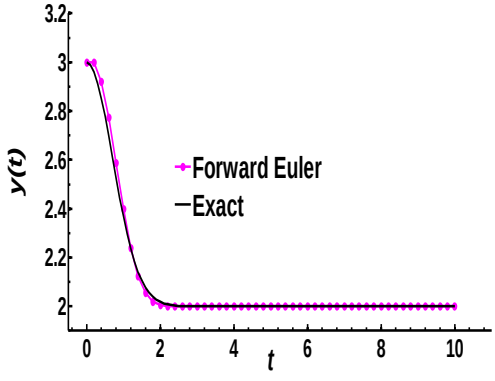


Figure 2: Comparison of FoEM with exact solution for (3.2) for step size Δt=0.2

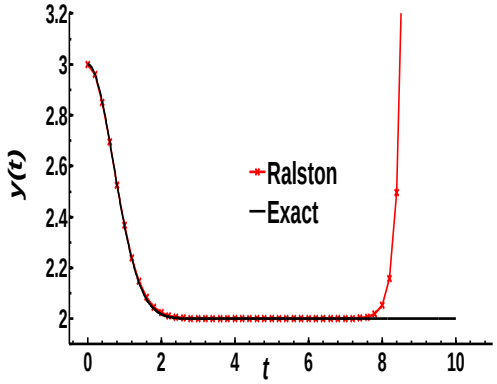


Figure 3: Comparison of Ralston's method with exact solution for (3.2) for step size Δt=0.2

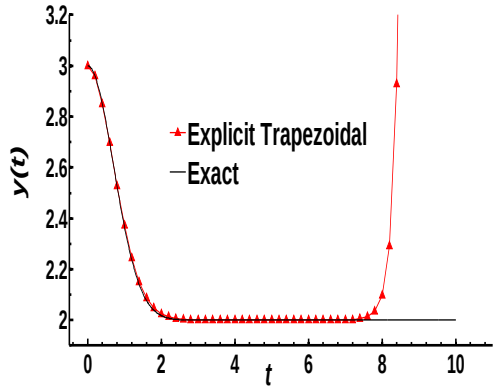


Figure 4: Comparison of Explicit Trapezoidal's method with exact solution for (3.2) for step size Δt=0.2

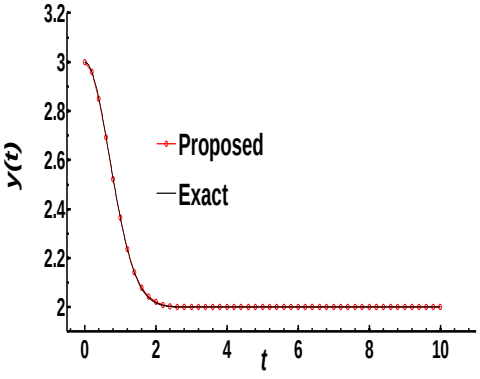


Figure 5. Comparison of Proposed Integrator with exact solution for (3.2) for step size Δt=0.2

From table 1, the ratio of the numerical errors at t=10 is:

$$\text{Ratio} = \frac{e(\Delta t = 0.05)}{e(\Delta t = 0.025)} = \frac{1.4010e-05}{3.5863e-06} = 3.9065 \quad (3.3)$$

Equation (3.3) declares the new iterative integrator to be of second order. This value (3.9065) is not exactly equal to the analytical value (4) due to finite step size

but the same can be achieved if $\Delta t \to 0$. Another feature depicted by the last column of table 1 is that the proposed method is convergent as by decreasing step size; the numerical solution is approaching analytical one. Finally, absolute percentile error involved in the proposed method is less than other methods shown in table 1. Comparison of absolute errors through figure 1 clearly reveals that the errors introduced by proposed method are always less than errors produced by other integrators. It can be observed from figures 2 to 5 that the proposed integrator is convergent and stable even if considerably large step size $(\Delta t = 0.2)$ is chosen for the given range. On the contrary, Explicit Trapezoidal and Ralston's methods pose instability as portrayed by both first row of table 2 and figures 3–4. Further FoEM also shows instabilities as the domain exceeds from the range $0 \le t \le 10$ until and unless considerably small step size is taken. In general; FoEM is of first order and does not perform better than Explicit Trapezoidal Rule, Ralston's Method, and Proposed Integrator.

The integrator suggested by (Sunday, *et al*., 2012) involves computation of four derivatives at given initial point whereas the proposed integrator does not need such computations at the cost of less accuracy. Furthermore, the method proposed can be applied to Cauchy problems defined at any interval but the method of (Sunday, *et al*., 2012) is not applicable for Cauchy problems defined at $t = 0$ as recommended by corresponding author of the paper when contacted via email[1].

4. CONCLUSION

In this paper, we have devised a new iterative integrator for Cauchy problems and the obtained results are compared with other methods as shown in table 1. Both linear and non–linear model problems are considered that revealed the efficiency of the integrator on comparison. It is found that the proposed integrator has shown better approximation to the exact solution even for larger step sizes. It is evident in figures 2 to 5, that the new integrator performs much better for systems with oscillatory and exponentially decaying solutions. Table 2 shows that maximum absolute error is significantly larger for three methods considered when compared with proposed integrator for integration steps equal to 50. Also, the newly devised integrator is found to be more accurate among the three methods as apparent by above shown computations. The condition for convergence and stability is achieved.

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