SindhUniv. Res. Jour. (Sci. Ser.) Vol. 52 (01) 37-40 (2020)

http://doi.org/10.26692/sujo/2020.03.06

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

A Modified Derivative-BasedScheme for the Riemann-Stieltjes Integral

K. MEMON[++], M. M. SHAIKH*, M. S. CHANDIO, A. W. SHAIKH

Institute of Mathematics and Computer Science, University of Sindh, Jamshoro

Received 21st October 2019 and Revised 5th March 2020

Abstract: In this research paper, a new efficient midpoint derivative-based quadrature scheme of trapezoid-type is developed for the approximation of the Riemann-Stieltjes integral. The proposed quadrature scheme is an efficient modification of Zhao *etal*. midpoint derivative-based scheme for the Riemann-Stieltjes integral. The performance of proposed scheme, Zhao *etal*. scheme and the original Trapezoid scheme for the Riemann-Stieltjesintegral are compared using MATLAB. Ithas been noticed that the numerical error is smaller using proposed scheme in comparison to other schemes.

Keywords: Quadrature rule, Riemann-Stieltjes integral, Trapezoidal rule.

1. INTRODUCTION

The basic problem in numerical integration is to compute an approximate value of a definite integral. In many engineering applications one have to calculate the area which is bounded by the curve of the function.Analytical solution for integral I(*f*) = $\int_a^b f(x)\,dx$ is not always available. For instance, the integrand*f*(*x*) may be defined in exponential and/or trigonometric functions with nonlinear argument such as $-e^{x^2}$ and $\sin x^2$, which cannot be integrated analytically. Approximating the integral I(*f*) numerically is referred as numerical integration, and the method by which approximation is obtained is called a quadrature rule. The study of Riemann-Stieltjes RS-integral: $\int_a^b f(x)\,d\alpha(x)$, where f is called the integrand, α is called the integrator, plays an important role in mathematics (Burden, 2011). RS-integrals are used in various areas of mathematics such as: Statistics and probability theory, Complex analysis, Functional analysis, Operator theory and others.

Most of the work on numerical integration has-been done for the Riemann integral. However a few studies have also focused on approximating the RS-integral. (Mercer, 2008) presented the extension of Trapezoidal rule for the RS-integral, and discussed the use of Hadamard integral inequality in the same context. He also used the idea of relative convexity and presented the work of midpoint and Simpson's rules for the RS-integral (Mercer, 2012). (Zhao and Li, 2013) presented a new family of closed Newton-Cotes quadrature rules for Riemann integral which uses the derivative value at the midpoint. (Zhao *et al.*, 2014) introduced the midpoint derivative-based trapezoid rule for the RS-integral in which midpoint is used for computing the function derivative. (Shaikh *et al.*, 2016) modified (Zhao and Li, 2013) midpoint derivative-based Simpson's 3/8th rule by a new four-point closed quadrature rule, and used second order midpoint derivative in each strip of integration for the comparison of fourth order derivative. (Shaikh, 2019) Compared the polynomial collocation method with uniformly spaced quadrature method for the solution of integral equations. (Bhatti*etal*., 2019) modified a numerical scheme developed by (Amanat, 2015) and proposed a new numerical integration scheme.In this paper, a new midpoint derivative-based scheme of trapezoid type for the RS-integral (Zhao *et al*., 2014) has been modified.

2. MATERIALS AND METHODS

2.1. General formulation of quadrature rules for the Riemann integral

General integration formula for evaluation of a definite integral over a finite interval [*a*, *b*] is expressed in(Burden, 2011) as follow:

$$I(f;a,b)=\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \quad (1)$$

where there exist$n$+1 distinct integration points at x_0, x_1, …,x_n within the interval [*a*, *b*] and *n*+1 weights w_i, i=0, 1, 2, ..., *n*. If the integration points are uniformly spaced over the interval, and are defined as:

$x_i = a+ih$, where $h=\dfrac{b-a}{n}$.

[++]Corresponding Author: Kashif Memon, Email: memonkashif.84@usindh.edu.pk

* Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro. (Mujtaba.shaikh@faculty.muet.edu.pk)

2.2. Original Trapezoid (T) Scheme for the RS-integral

In (Zhao *et al.*, 2015), the classical Trapezoid rule of the Riemann integrals was extended for the approximation of RS-integral. Theresulting derivative-free Trapezoid rule for RS- integral as in (Zhao *et al.*, 2015) with local error term is:

$$\int_a^b f(x)\,dg = \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b) - \frac{(b-a)^3}{12} f''(\xi) g'(\eta) \qquad (2)$$

where $\xi, \eta \in (a,b)$.

The composite scheme of original T for RS-integral which is referred here as CT given in(Zhao *et al.*, 2015) and is defined as:

$$\int_a^b f(t)\,dg = \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b) - \frac{(b-a)^3}{12n^2} f''(\mu) g'(\eta) \qquad (3)$$

where $\mu, \eta \in (a,b)$.

This scheme (3) is used to minimize the local error at every approximation.

2.3. Midpoint Derivative-Based Trapezoid (ZT) Scheme for the RS-integral

(Zhao *et al.*, 2014) derived the midpoint derivative-based Trapezoidal rule for the RS- integral, which is:

$$\int_a^b f(x)\,dg = \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b) + \left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right) f''(c), \qquad (4)$$

where

$$c = \frac{\left(-2b^2 + a^2 - ab\right)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}.$$

The precision of this method is three.The error term R[f] of this method was obtained as:

$$\left(\begin{array}{l} \dfrac{a^3 + ab^2 + a^2 b - 3b^3 + 6(b-a)c^2}{24}\int_a^b g(t)dt + \dfrac{b^2 - c^2}{2}\int_a^b\int_a^t g(x)dxdt \\ -b\int_a^b\int_a^t\int_a^y g(x)dxdt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \end{array}\right) f^{(4)}(\xi),$$

where $\xi \in (a,b)$.

2.4. Proposed Modified Midpoint Derivative-Based Trapezoid (MZT) Scheme for the RS- integral

In scheme (4), substituting$g(t) = t$, the value of c, which is givenin (4) is not reduced to $\frac{a+b}{2}$ as claimed by the author (Zhao *et al.*, 2014), rather it reduces to

$$c = \frac{(a+b)^3}{2(a-b)^2}.$$

Here, the work of (Zhao *et al.*, 2014) is modified which is referred here as MZT and is given below:

$$\int_a^b f(x)\,dg \approx \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b) + \left(\int_a^b\int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right) f''(c), \qquad (5)$$

where

$$c = \frac{\left(-2b^2 + a^2 + ab\right)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}.$$

Now, if $g(t) = t$ is substituted in (5), the value of $c$reduces to $c = \frac{a+b}{2}$.

The precision of this method is also three. The error term $R_{ZT}[f]$of this method has been obtained as:

$$\begin{pmatrix} \dfrac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\displaystyle\int_a^b g(t)dt \\ +\dfrac{b^2-c^2}{2}\displaystyle\int_a^b\int_a^t g(x)dxdt - b\int_a^b\int_a^t\int_a^y g(x)dxdt \\ +\displaystyle\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \end{pmatrix} f^{(4)}(\xi),$$

where $\xi \in (a,b)$.

In this research paper, the composite form of scheme (4) of (Zhao *et al.*, 2014) midpoint derivative-based trapezoidal rule (ZCT) for the RS-integral has been derived as:

$$\begin{aligned}\int_a^b f(t)\,dg \approx & \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) \qquad (6)\\ & +\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k)\\ & +\left[g(b)-\frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b)\\ & +\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right] f''(c_k),\end{aligned}$$

where

$$c_k = \frac{\left(-2x_k^2+x_{k-1}^2-x_{k-1}x_k\right)\int_{x_{k-1}}^{x_k} g(t)dt + 6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - 6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}.$$

The error term $R_{ZCT}[f]$ of (6) is:

$$n\begin{pmatrix} \dfrac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\displaystyle\int_a^b g(t)dt + \dfrac{b^2-c^2}{2}\int_a^b\int_a^t g(x)dxdt \\ -b\displaystyle\int_a^b\int_a^t\int_a^y g(x)dxdt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \end{pmatrix} f^{(4)}(\mu)g'(\eta)$$

where $\mu,\eta \in (a,b)$.

The composite form of MZT (5), is referred as MZCT, is described as:

$$\begin{aligned}\int_a^b f(t)\,dg \approx & \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) \qquad (7)\\ & +\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k)\\ & +\left[g(b)-\frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b)\\ & +\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{h}{2}\int_{x_{k-1}}^{x_k} g(t)dt\right] f''(c_k),\end{aligned}$$

$$c_k = \frac{\left(-2x_k^2+x_{k-1}^2+x_{k-1}x_k\right)\int_{x_{k-1}}^{x_k} g(t)dt + 6b\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - 6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt}{6\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt - \frac{3(x_k-x_{k-1})}{n}\int_{x_{k-1}}^{x_k} g(t)dt}.$$

The error term $R_{MZCT}[f]$ of (7) is:

$$n\begin{pmatrix} \dfrac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\displaystyle\int_a^b g(t)dt \\ +\dfrac{b^2-c^2}{2}\displaystyle\int_a^b\int_a^t g(x)dxdt - b\int_a^b\int_a^t\int_a^y g(x)dxdt \\ +\displaystyle\int_a^b\int_a^t\int_a^z\int_a^y g(x)dxdydzdt \end{pmatrix} f^{(4)}(\mu)g'(\eta)$$

where $\mu,\eta \in (a,b)$.

3. RESULTS AND DISCUSSION

The efficiency of the proposed derivative-based Trapezoid-type quadrature scheme MZCT for the RS-integral is compared against the existing CT and ZCT schemes in terms of absolute error distributions with increasing number of strips of integration. It has been noticed that in the previous studies (Mercer, 2012) and(Zhao *et al.*, 2014) on the quadrature schemes for RS-integral, the numerical work has never been conducted, whereas the present study also confirms the validity of theoretical results as well, using the following two examples. The exact approximations up to 16 decimal placesaccuracy of the two test integrals taken from (Protter, 1977) and (Bartle, 1964)are mentioned alongside the examples, which have been computed using MATLAB software. All the results were compared in Intel (R) Core (TM) Laptop with RAM 4.00GB and processing speed of 0.80GHz-1.00GHz. For numerical results, double precision arithmetic is used.

Example-1

$$\int_{3.5}^{4.5} \sin 5x \; d(\cos x) = 0.227676016130689$$

Example-2

$$\int_{5}^{6} \sin x \; d\left(x^3\right) = 59.655908136641912$$

The absolute error is defined as:

Absolute Error = | Exact value – Approximate value |

(Fig.1), represents the comparison of threeschemes in terms of absolute errors versus the number of strips ranging from 1 to 50.

(Fig.2) shows the comparison of threeschemes in terms of absolute errors versus the number of strips which equally divide the interval of integration into the sequence of subintervals ranging from 5, 10, …, 200.

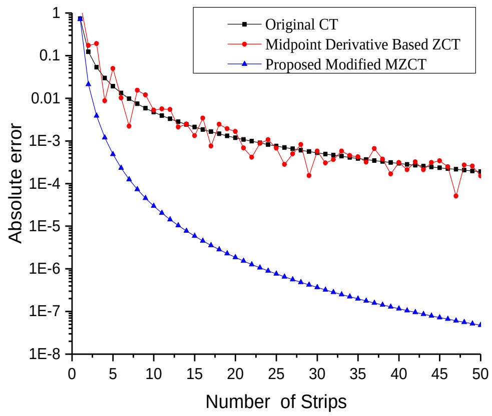

Fig. 1. Absolute error distributions for Example 1 versus number of strips.

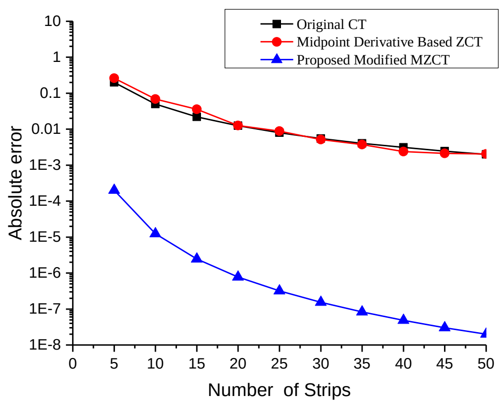

Fig. 2. Absolute error distributions for Example 2 versus number of strips.

It has been also noticed in Figure 1, the proposed scheme MZCT has converged to require level of accuracy. In contrast, ZCT scheme shows fluctuation while converging to required level of accuracy.

It has been observed in Fig. 1 and 2, that the absolute errors obtained using the proposed MZCT scheme are much smaller in comparison to CT and ZCT schemes throughout the used range of number of strips in Examples 1 and 2.

4. CONCLUSION

The new efficient midpoint derivative-based scheme of trapezoid-type is proposed for the RS-integral. The proposed scheme is a modification of Zhao *et. al.* mid-point derivative based version of trapezoid-typescheme. The proposed MZTscheme minimizes the errors and improves the accuracy level in comparison of two existing schemes. Thus our proposed scheme has also maintained the numerical stability.

In future, the proposed scheme may be extended for other quadrature schemes.

REFERENCES:

Amanat U., (2015), "Numerical Integration and a Proposed Rule", American Journal of Engineering Research(AJER),(4): 120-123.

Bartle, R. G., (1964), "The elements of real analysis", John Wiley & Sons.

Bhatti, A. A., M. S. Chandio, R. A. Memon and M. M. Shaikh, (2019), "A Modified Algorithm for Reduction of Error in Combined Numerical Integration", Sindh University Research Journal SURJ (Science Series) 51(4): 745-750.

Burden, R. L., and J. D. Faires, (2011) "Numerical Analysis", Brooks/Cole, Boston, Mass, USA, 9th edition.

Mercer, P. R., (2008), "Hadamard's inequality and Trapezoid rules for the Riemann-Stieltjes integral", Journal of Mathematica Analysis and Applications, 344: 921-926.

Mercer, P. R., (2012), "Relative convexity and quadrature rules for the Riemann-Stieltjes integral", Journal of Mathematica inequality, 6: 65-68.

Protter, M. H. and C. B. Morrey, (1977), "A First Course in Real Analysis". Springer, New York, NY.

Shaikh, M. M., M. S. Chandio and A. S. Soomro, (2016), "A Modified Four-point Closed Mid-point Derivative Based Quadrature Rule for Numerical Integration", Sindh University Research Journal-SURJ (Science Series) 48(2): 389-392.

Shaikh, M. M., (2019), "Analysis of Polynomial Collocation and Uniformly Spaced Quadrature Methods for Second Kind Linear Fredholm Integral Equations – A Comparison". Turkish Journal of Analysis and Number Theory, 7(4) 91-97. doi: 10.12691/tjant-7-4-1.

Zhao, W., and H. Li, (2013) "Midpoint Derivative-Based Closed Newton-Cotes Quadrature", Abstract And Applied Analysis, Article ID 492507.

Zhao, W., Z. Zhang, and Z. Ye, (2014), "Midpoint Derivative-Based Trapezoid Rule for the Riemann-Stieltjes Integral", Italian Journal of Pure and Applied Mathematics, 33: 369-376.

Zhao, W., Z. Zhang, and Z. Ye, (2015), "Composite Trapezoid rule for the Riemann-Stieltjes Integral and its Richardson Extrapolation Formula", Italian Journal of Pure and Applied Mathematics,35: 311-318.