



Transient Analysis of Phase Type Hysteresis system with Dual Arrival Rate Through Matrix Geometric Method

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Abstract: Transient analysis is the efficient way to observe the system behavior with respect to time when system not reached in steady state. In this paper, we analyze the behavior of phase type hysteresis system with dual arrival rate in transient state. Service process of system is modeled with hypoexponential and hyperexponential phase type distribution and solved through vector domain technique called matrix geometric method. In this system, arrival rate of customer switched between two rates which depends on the system length. We analyze the effect of dual arrival rate and internal thresholds on the system length in the transient state by using hypoexponential and hyperexponential distributions as a service process.

Keyword: Transient, Phase type distribution, Hysteresis, Dual arrival rate, Matrix geometric method

1. INTRODUCTION

Modelling techniques based on analytical methods play an important role in evaluating the performance of computer and communication systems (El-Rayes, 1999). Generally, performance modelling is based on the queueing theory which uses the queue phenomenon. The Markov chain is constructed from the queueing model and then solved through various numerical solution techniques. There are two main classes of numerical methods to solve the Markov chain, direct methods and iterative methods. Direct methods utilize the parameter matrix and thereby commonly used methods are Gauss elimination and Grassman's algorithms. The iterative methods depend on the selection of the initial value. Whereas the commonly used iteration methods are the power method, Jacobi's method (Ramswami, 1996), Gauss seidel method and successive over relaxation (SOR) method. As the complexity of these systems increases, the efficiency and applicability of the traditional analytical modelling techniques become inadequate (Stewart, 1994).

Transient analysis is the important performance analysis where the system not reached in its steady state. Most of the system transient analysis is done in scalar domain by utilizing the SOR and Gauss seidel method with the combination of the scalar form Runge-Kutta method (Riska, 2002). These solution techniques become very complex as the system states grows and it's become difficult to obtain the closed form solution. The vector domain transient analysis efficiently be done through the vector domain method which utilizing the special characteristics of the structured Markov chain.

Structured Markov chains have a special structure in their transition matrix, which can be used to solve the system more easily and efficiently through an

efficient solution method based on vector form analysis instead of the scalar analysis (Osogami, 2006). The method which utilizes the structure of the transition rate matrices of the structured Markov chain is known as matrix geometric method which is an analytical tools to analyze the system performance through efficient algorithms. The main advantage of this method is that it has a direct probabilistic interpretation which is given in the matrices and it also does not use the theory of Eigenvalues.

Transient analysis of queueing systems with phase-type distributions can be solved easily and efficiently as compared to other numerical solution techniques as a quasi birth death (QBD) process through the matrix geometric method (Lang, 1996, VanHoudt, 2004).

In this paper, we analyze the transient behavior of phase type hysteresis system using hypoexponential and hyperexponential distributions with dual arrival rate through matrix geometric method.

Phase Type Distribution

A phase-type distribution describes the time to absorption of a finite Markov chain in continuous time, where there is one absorbing state and the stochastic process starts in a transient state (Bodrog, 2007). Erlang, Hypoexponential, Hyperexponential and Coaxian are the special cases of phase type distributions. The Erlang distribution has two parameters, the phase an integer $k > 0$ and the rate $r > 0$ and it is denoted as $E(k, r)$. The hypoexponential distribution is the generalization of the Erlang distribution by having the different rates for each transition. The Coxian distribution is the generalization of the hypoexponential distribution in which state can be reached to the absorbing state from any phase.

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2) **MATRIX GEOMETRIC METHOD**

Matrix geometric method is the efficient and simplest solution technique for obtaining the steady state probability vector of the Markov chain. The technique is applied to only those stochastic processes, whose steady state probability vector exists a geometric relation. The method consists on following steps following steps

1. Construction of structured Markov chain
2. Develop an infinitesimal generator matrix from structured Markov chain
3. Partitioning of levels and phases in structured Markov chain and infinitesimal generator matrix.
4. Obtain submatrices
5. Compute rate matrix R through submatrices
6. Obtain stationary probability vector
7. Compute performance measures through R and stationary probability vector

Phase Type Hysteresis System with Dual Arrival Rate

The hysteresis system with dual arrival rate is shown in (Fig. 1), in which arrival rate depends on the number of the customers in the system. The system has two arrival rates one is called normal and other is reduced. Both arrival rates are controlled through the number of customers presents in the system.

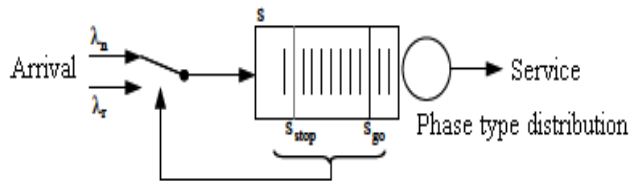


Fig. 1. Phase Type Hysteresis System with Dual Arrival Rate

Initially customers arrives with normal rate λ_n up to the threshold S_{stop} . When the number of customer reached on S_{stop} , the normal arrival rate will be switched and reduced to the λ_r , which protects the system become full. The arrival rate will be switched back to the normal arrival rate λ_n after reaching the number of the customers to the threshold S_{go} . Phase-type distributions are used as a service process in a hysteresis system. The following are the systems which use the Erlang and Coxian distributions as a service process.

Hysteresis System with two stage hypoexponential distribution

The service process in a hysteresis system with dual arrival rate is a two-phase hypoexponential

distribution and the arrival process follows a Markovian process as shown in (Fig. 2). The service rate of

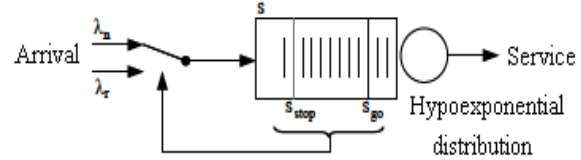


Fig. 2. Hypoexponential Distribution Type Hysteresis System with Dual Arrival Rate

each phase is μ_1 and μ_2 , respectively. The construction of structured Markov chain of this system is shown in (Fig. 3). The system state of structured Markov chain is represented by $(i/j/a)$, where i represent the number of customers in the system, j represents the phase of service and a represents the arrival rate.

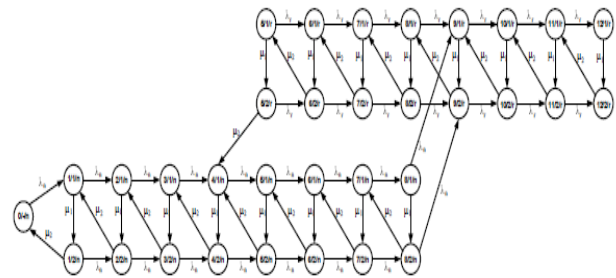


Fig. 3. Structured Markov chain

The finitesimal generator matrix can be constructed from structured Markov chain by arranging the system states in a lexicographical order. The resultant block finitesimal generator matrix has QBD form in which system is divided into three subsystems, first system consist on from idle system to the first threshold, second system contains system states from first threshold to the second threshold and final system contains the system states from second threshold to the final system capacity. The block finitesimal generator matrix is shown in Equation 1 and it consists of submatrices which represents the three sub systems. The initial and final boundaries of the system defined by the boundary matrices B_0 and B_9 . The first sub system defines by the submatrices B_1, B_2, B_3, A_1, A_2 and A_3 . The submatrices $B_4, B_5, B_6, \bar{A}_1, \bar{A}_2, \bar{A}_3$ represents the second subsystem and the third subsystem represent by the submatrices $B_7, B_8, \hat{A}_1, \hat{A}_2, \hat{A}_3$.

$$\begin{matrix}
 & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix} \\
 \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix} & \left(\begin{array}{cccccccccccc}
 B_0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 B_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & A_2 & B_3 & B_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & B_5 & A_1 & \bar{A}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & A_2 & \bar{A}_1 & \bar{A}_0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_2 & B_6 & B_7 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_8 & \hat{A}_1 & \hat{A}_0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{A}_2 & \hat{A}_1 & \hat{A}_0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{A}_2 & \hat{A}_1 & \hat{A}_0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{A}_2 & B_9
 \end{array} \right)
 \end{matrix} \quad (1)$$

The boundary and repetitive submatrices of each system are represented in Equation 2.

$$\begin{aligned}
 B_0 &= [-\lambda_0] & B_1 &= [-\lambda_0 \quad 0] & B_2 &= \begin{bmatrix} 0 \\ \mu_2 \end{bmatrix} & A_0 &= \begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix} & A_1 &= \begin{bmatrix} a & \mu_1 \\ 0 & b \end{bmatrix} \\
 A_2 &= \begin{bmatrix} 0 & 0 \\ \mu_2 & 0 \end{bmatrix} & B_3 &= \begin{bmatrix} a & \mu_1 \\ 0 & b \end{bmatrix} & B_4 &= \begin{bmatrix} \lambda_n & 0 & 0 & 0 \\ 0 & \lambda_n & 0 & 0 \end{bmatrix} & B_5 &= \begin{bmatrix} 0 & 0 \\ \mu_2 & 0 \\ 0 & 0 \\ \mu_2 & 0 \end{bmatrix} \\
 \bar{A}_0 &= \begin{bmatrix} \lambda_n & 0 & 0 & 0 \\ 0 & \lambda_n & 0 & 0 \\ 0 & 0 & \lambda_n & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} & \bar{A}_1 &= \begin{bmatrix} a & \mu_1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & \mu_1 \\ 0 & 0 & 0 & d \end{bmatrix} & \bar{A}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 \end{bmatrix} & B_6 &= \begin{bmatrix} a & \mu_1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & \mu_1 \\ 0 & 0 & 0 & d \end{bmatrix} \\
 B_7 &= \begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \\ \lambda_r & 0 \\ 0 & \lambda_r \end{bmatrix} & B_8 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \end{bmatrix} & \hat{A}_0 &= \begin{bmatrix} \lambda_r & 0 \\ 0 & \lambda_r \end{bmatrix} & \hat{A}_1 &= \begin{bmatrix} c & \mu_1 \\ 0 & d \end{bmatrix} & \hat{A}_2 &= \begin{bmatrix} 0 & 0 \\ \mu_2 & \lambda_r \end{bmatrix} & B_9 &= \begin{bmatrix} -\mu_1 & \mu_1 \\ 0 & -\mu_2 \end{bmatrix} \\
 a &= (-\lambda_n + \mu_1) & b &= (-\lambda_n + \mu_2) & c &= (-\lambda_r + \mu_1) & d &= (-\lambda_n + \mu_2)
 \end{aligned} \tag{2}$$

B. Hysteresis System with two stage hyperexponential distr

The hysteresis system with two-phase hyperexponential distribution as a service process is shown in (Fig. 4).

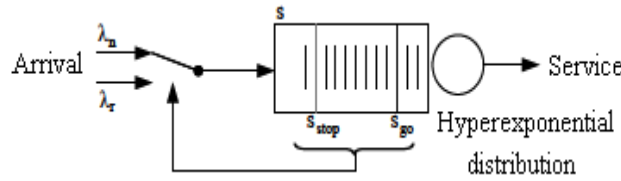


Fig. 4. Hyperexponential Distribution Type Hysteresis System with Dual Arrival Rate

The structured Markov chain of the system is shown in (Fig. 5), where service of each customer follows a hyperexponential distribution.

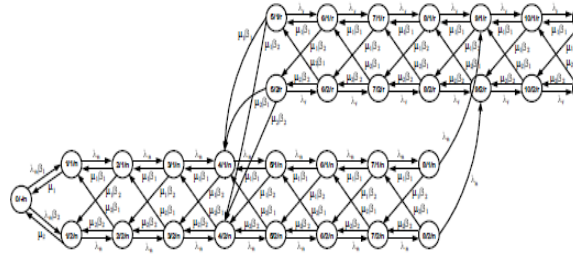


Fig. 5. Structured Markov chain

The block finite generator matrix has the same form as hysteresis system with hypoexponential distribution but the submatrices are different due to the hyperexponential distribution where at each state in the system, after receiving service the system state will be back in first stage of service or second stage of service. The submatrices of the system represented in Equation 3.

$$\begin{aligned}
 B_0 &= [-\lambda_n] & B_1 &= [-\lambda_n \beta_1 \quad -\lambda_n \beta_1] & B_2 &= \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} & A_0 &= \begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix} & A_1 &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \\
 A_2 &= \begin{bmatrix} \mu_1 \beta_1 & \mu_1 \beta_2 \\ \mu_2 \beta_1 & \mu_2 \beta_2 \end{bmatrix} & B_3 &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} & B_4 &= \begin{bmatrix} \lambda_n & 0 & 0 & 0 \\ 0 & \lambda_n & 0 & 0 \end{bmatrix} & B_5 &= \begin{bmatrix} \mu_1 \beta_1 & \mu_1 \beta_2 \\ \mu_2 \beta_1 & \mu_2 \beta_2 \\ \mu_1 \beta_1 & \mu_1 \beta_2 \\ \mu_2 \beta_1 & \mu_2 \beta_2 \end{bmatrix} \\
 \bar{A}_0 &= \begin{bmatrix} \lambda_n & 0 & 0 & 0 \\ 0 & \lambda_n & 0 & 0 \\ 0 & 0 & \lambda_n & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} & \bar{A}_1 &= \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} & \bar{A}_2 &= \begin{bmatrix} \mu_1 \beta_1 & \mu_1 \beta_2 & 0 & 0 \\ \mu_2 \beta_1 & \mu_2 \beta_2 & 0 & 0 \\ 0 & 0 & \mu_1 \beta_1 & \mu_1 \beta_2 \\ 0 & 0 & \mu_2 \beta_1 & \mu_2 \beta_2 \end{bmatrix} & B_6 &= \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \\
 B_7 &= \begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \\ \lambda_r & 0 \\ 0 & \lambda_r \end{bmatrix} & B_8 &= \begin{bmatrix} 0 & 0 & \mu_1 \beta_1 & \mu_1 \beta_2 \\ 0 & 0 & \mu_2 \beta_1 & \mu_2 \beta_2 \end{bmatrix} & \hat{A}_0 &= \begin{bmatrix} \lambda_r & 0 \\ 0 & \lambda_r \end{bmatrix} & \hat{A}_1 &= \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} & \hat{A}_2 &= \begin{bmatrix} \mu_1 \beta_1 & \mu_1 \beta_2 \\ \mu_2 \beta_1 & \mu_2 \beta_2 \end{bmatrix} & B_9 &= \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix} \\
 a &= (-\lambda_n + \mu_1) & b &= (-\lambda_n + \mu_2) & c &= (-\lambda_r + \mu_1) & d &= (-\lambda_n + \mu_2)
 \end{aligned} \tag{3}$$

3. RESULTS AND DISCUSSION

The analytical program for matrix geometric method along with vector form Runge-Kutta procedure is written in VC++ for transient analysis of the phase type hysteresis system with dual arrival rate. Transient analysis program is also developed in Matlab for matrix geometric method.

The transient analysis of the phase type hysteresis system with dual arrival rate through matrix geometric method is given in (Fig. 6) and (Fig.7) for various system sizes, S_{go} and S_{stop} thresholds. (Fig. 6) shows the transient behavior of hysteresis system with hyperexponential distribution against varying the time. It shows that mean number of customers in the system is not effected till time $t=10$, it means system is idle till time is 10, and then there is an abrupt increase in the system length. At time $t = 10$, mean system length increases to the second threshold S_{stop} and then decreases near to the first threshold S_{go} . Here, it is clearly seen that mean system length is again increase before reaching to the first S_{go} threshold. The system will not reach its maximum capacity and it length oscillate between the first S_{go} and Second S_{stop} thresholds.

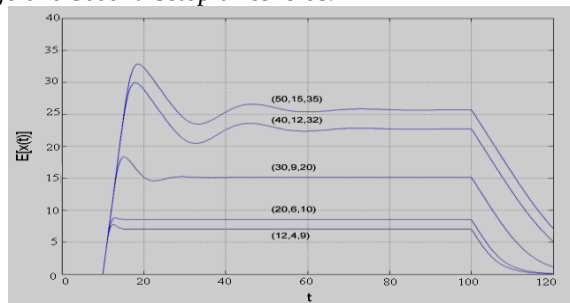


Fig. 6: Mean number in the system: Transient analysis with hyperexponential distribution, $\lambda_n = 6$, $\lambda_r = 0.1$, $\mu = 1$, $cv = 1.5$

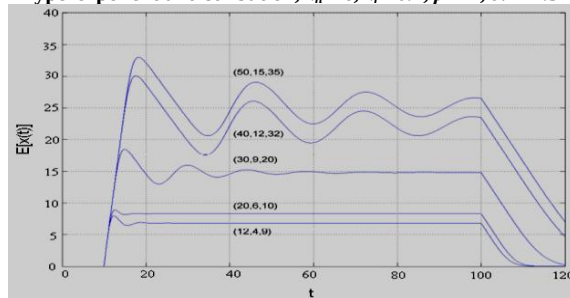


Fig. 7: Mean number in the system: Transient analysis with hypoexponential distribution, $\lambda_n = 6$, $\lambda_r = 0.1$, $\mu = 1$, $cv = 0.71$

Here, it is clearly observed that the difference between the S_{go} and S_{stop} thresholds will effect on the oscillation. The graph clearly shows that the smaller oscillation occurs due to the smaller difference between the S_{go} and S_{stop} thresholds and the system will be stabilized soon and the system has more oscillations when there is larger difference between the S_{go} and S_{stop} thresholds. (Fig. 7) shows the mean number of customers in the system against time for the transient analysis. Here,

it is observed that the same behavior for the system with hypoexponential distribution is seen as observed in (Fig. 6). It is clearly seen that the system with hypoexponential distribution has more oscillations as compared to the other system. The hyperexponential system has smaller oscillation and stabilizes soon as compared to the hypoexponential system.

4. CONCLUSION

In this paper, we used a phase type queueing approach to model the hysteresis system with dual arrival rate. The system is solved using internal threshold system in which the service process is modeled through a continuous time hypoexponential and hyperexponential phase type distributions.

The structured Markov chains are constructed and matrix geometric method is used to analyze the transient behavior of these systems. The mean system length is analyzed in transient state and observed that the system is oscillates between two thresholds which protect the system for fully occupied. An obtained analytical result shows that the system with hypoexponential distribution has more oscillations as compared to the hyperexponential system.

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