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Flow Time Analysis of Phase Type Queueing System Using Matrix Geometric Method

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Abstract: Mean flow time analysis is the efficient way to trace the flow of customers in various systems. In this paper we develop three different quasi birth death systems employing a vector form phase type distribution to analyze the flow time of customers in the system. These systems are solved through a special vector domain analytical technique called matrix geometric method. The resulting submatrices are used to obtain the mean flow time of these systems. We analyze flow time for different QBD systems having different and same phase type distributions as an arrival or service or both processes.

Keywords: Queueing systems, Phase type distribution, QBD process, Matrix geometric method

1. INTRODUCTION

Quasi Birth and Death (QBD) process is an efficient way to represent the system behavior and its characteristics. The systems including telecommunication system, computer communication system etc can be modeled and analyzed as a QBD process (Ramswami, 1996). Mainly the communication systems behaves like a QBD process in nature and need many stages to completely enter in the system or leave the system after served (Stewart, 1994). These types of the systems can efficiently be modeled as phase type QBD process in which the arrival or service or both arrival and service processes follows a phase type distribution. In some systems arrival or service requires to pass all stages to fully enter in the system or receiving full service. Similarly some of the system does not require passing every stage for the arrival and service but they can enter in or leave the system from any stage. Mostly these types of the systems are analyzed in scalar domain where resulting differential equations are difficult to handle and to obtain the closed form solution. These systems can efficiently be solved in vector domain by utilizing the characteristics of a structured Markov chain (El-Rayes, 1999). An efficient numerical method called matrix geometric method can be used to analyze the system by utilizing the characteristics of the structured Markov chain and its resulting transition matrix. The mean flow time of the customer in the system which modeled as phase type QBD process can be analyzed through matrix geometric method (Latouche, 1986). (Neuts, 198, Neuts, 1980). In (Latouche, 1993) parallel systems are analyzed as QBD and solved through invariant matrices. The modeling of the risk in the system through phase type distribution is discussed in (Willmort, 2011).

2. MATERIALS AND METHODS

1) *Phase-type distribution*

A phase-type distribution describes the time to absorption of a finite Markov chain in continuous time, where there is one absorbing state and the stochastic process starts in a transient state. A row vector α of size n is associated with the underlying Markov chain of a PH distribution and represents the initial probability vector for each of its transient states. The transition rate matrix of the process can be written as:

$$Q = \begin{pmatrix} T & \mathrm{T}^0 \\ 0 & 0 \end{pmatrix}$$

Where T is an $n\times n$ matrix representing the transitions among the transient states and 0 is a column vector of size $n \times 1$ representing the transitions from the transient states to the absorbing state of the underlying Markov chain, i.e, T0 = -Te. The PH distribution is represented by the vector α and the matrix T.

2) Kronecker Product

The Kronecker product is an operation of two matrices of an arbitrary size resulting in a block matrix. The Kronecker product of two matrices A and B of dimension $m \times n$ is

$$A \otimes \mathrm{B} = \begin{bmatrix} a_{11}\mathrm{B} & a_{12}\mathrm{B} & a_{13}\mathrm{B} & a_{14}\mathrm{B} & \cdots & a_{1n}\mathrm{B} \\ a_{21}\mathrm{B} & a_{22}\mathrm{B} & a_{23}\mathrm{B} & a_{24}\mathrm{B} & \cdots & a_{2n}\mathrm{B} \\ a_{31}\mathrm{B} & a_{32}\mathrm{B} & a_{33}\mathrm{B} & a_{34}\mathrm{B} & \cdots & a_{3n}\mathrm{B} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}\mathrm{B} & a_{m2}\mathrm{B} & a_{m3}\mathrm{B} & a_{m4}\mathrm{B} & \cdots & a_{mn}\mathrm{B} \end{bmatrix}$$

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3) Phase type QBD queueing system

In this paper we develop a three different phase type QBD models based on the modeling of arrival or service or both processes as a phase type distribution ([Shah, Wajiha., 2010, Shah, Wajiha., *et. al.,* 2010).

B. Phase type distribution arrival process

An infinite queue in which arrival process follows the phase-type distribution and service is exponentially distributed.

Hyperexponential distribution with r stages Consider an infinite queue in which arrivals occurs according to the Hyperexponential distribution with r stages each with rate $\beta_i\lambda_i$, $i = 1, 2, \ldots, r$ and service is exponentially distributed with rate μ¸ is shown in (**Fig. 1**).

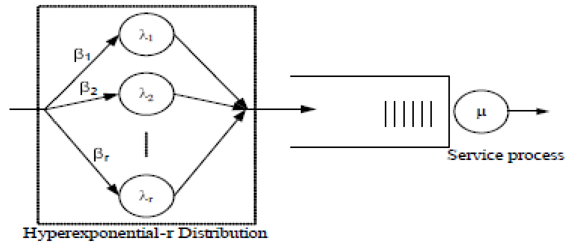
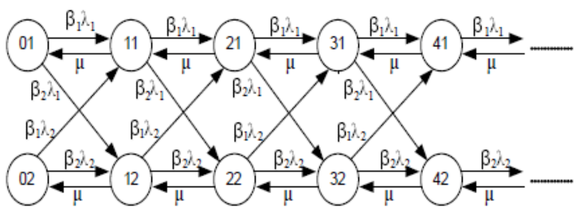


Fig. 1: **Queueing model of *Hyperr/M/*1**

The Markov chain of the *Hyperr/M/*1 is constructed as shown in (**Fig. 2a**).

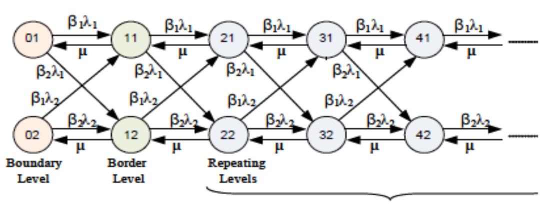


$$Q_{Hyper_2/M/1} =$$

$$\begin{array}{c} \\ 01 \\ 02 \\ 11 \\ 12 \\ 21 \\ 22 \\ 31 \\ 32 \\ \vdots \end{array}
\begin{array}{ccccccccc} 01 & 02 & 11 & 12 & 21 & 22 & 31 & 32 & \ldots \\
-\lambda_1 & 0 & \beta_1\lambda_1 & \beta_2\lambda_1 & 0 & 0 & & 00 & \ldots \\
0 & -\lambda_1 & \beta_1\lambda_2 & \beta_2\lambda_2 & 0 & 0 & 0 & \ldots & \\
\mu & 0 & a_1 & 0 & \beta_1\lambda_1 & \beta_2\lambda_1 & 0 & 0 & \ldots \\
0 & \mu & 0 & a_2 & \beta_1\lambda_2 & \beta_2\lambda_2 & 0 & 0 & \ldots \\
0 & 0 & \mu & 0 & a_1 & 0 & \beta_1\lambda_1 & \beta_2\lambda_1 & \ldots \\
0 & 0 & 0 & \mu & 00 & a_2 & \beta_1\lambda_2 & \beta_2\lambda_2 & \ldots \\
0 & 0 & 0 & 0 & \mu & a_1 & 0 & \ldots & \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & a_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

Fig. 2: a) Markov chain b) Infinitesimal generator matrix

The system state space is represented by a (ij), i represents the number of customers in the system and j represents the current phase of t he arrival. The infinitesimal generator matrix is constructed by arranging the states in a lexicographical order as shown in (**Fig. 2b**). Partitioning of Markov chain and infinitesimal generator matrix is given as shown in (**Fig. 3**).



$$Q_{Hyper_2/M/1} = \left(\begin{array}{cc|cc|cc|cc|c}
-\lambda_1 & 0 & \beta_1\lambda_1 & \beta_2\lambda_1 & 0 & 0 & 0 & 0 & \ldots \\
0 & -\lambda_2 & \beta_1\lambda_2 & \beta_2\lambda_2 & 0 & 0 & 0 & 0 & \ldots \\ \hline
\mu & 0 & a_1 & 0 & \beta_1\lambda_1 & \beta_2\lambda_1 & 0 & 0 & \ldots \\
0 & \mu & 0 & a_2 & \beta_1\lambda_2 & \beta_2\lambda_2 & 0 & 0 & \ldots \\ \hline
0 & 0 & \mu & 0 & a_1 & 0 & \beta_1\lambda_1 & \beta_2\lambda_1 & \ldots \\
0 & 0 & 0 & \mu & 0 & a_2 & \beta_1\lambda_2 & \beta_2\lambda_2 & \ldots \\ \hline
0 & 0 & 0 & 0 & \mu & 0 & a_1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & a_2 & \ldots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}\right)$$

Fig. 3: Partitioning of Markov chain and infinitesimal generator matrix

The block submatrices are constructed as shown in Equation 1.

$$B_{00} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_1 \end{bmatrix} \quad B_{01} = \begin{bmatrix} \beta_1\lambda_1 & \beta_2\lambda_1 \\ \beta_1\lambda_2 & \beta_2\lambda_2 \end{bmatrix} \quad B_{10} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \quad A_0 = \begin{bmatrix} \beta_1\lambda_1 & \beta_2\lambda_1 \\ \beta_1\lambda_2 & \beta_2\lambda_2 \end{bmatrix} \tag{1}$$

$$A_1 = \begin{bmatrix} -(\lambda_1+\mu) & 0 \\ 0 & -(\lambda_1+\mu) \end{bmatrix} \quad A_2 = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$$

Erlang distribution with r stages

Consider an infinite queue in which arrivals follows the Erlang-r distribution with rate r¸ and the service process is exponentially distributed with rate μ is shown in (**Fig. 4**).

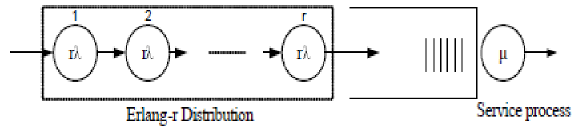
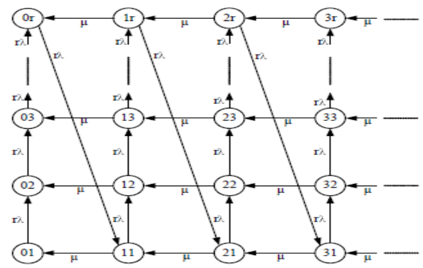


Fig. 4: **Queueing model of *Er/M/*1**

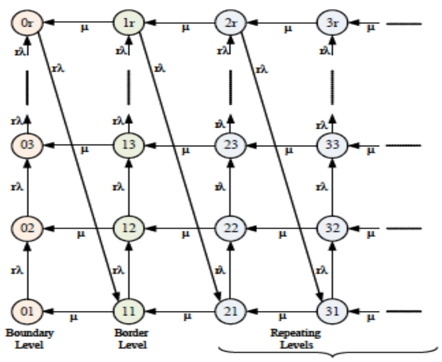
The Markov chain and its resulting infinitesimal generator matrix are shown in (**Fig. 5)**.



$$Q_{E_r/M/1} = \begin{array}{c} \\ 00 \\ 02 \\ \vdots \\ 0r \\ 11 \\ 12 \\ \vdots \\ 1r \\ 21 \\ 22 \\ \vdots \\ 2r \\ \vdots \end{array} \begin{array}{cccccccccccccc} 01 & 02 & \cdots & 0r & 11 & 12 & \cdots & 1r & 21 & 22 & \cdots & 2r & \cdots \\ -r\lambda & r\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & -r\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & -r\lambda & r\lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \mu & 0 & \cdots & 0 & a & r\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \mu & \cdots & 0 & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & \mu & 0 & 0 & \cdots & a & r\lambda & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \mu & 0 & \cdots & 0 & a & r\lambda & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \mu & \cdots & 0 & 0 & a & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \mu & 0 & 0 & \cdots & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Fig. 5: **Markov chain and infinitesimal generator matrix**

Partitioning of the levels in Markov chain and infinitesimal generator matrix is given as in (**Fig. 6)**.



$$Q_{E_r/M/1} = \left(\begin{array}{cccc|cccc|cccc|c} -r\lambda & r\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & -r\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & -r\lambda & r\lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \hline \mu & 0 & \cdots & 0 & a & r\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \mu & \cdots & 0 & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & \mu & 0 & 0 & \cdots & a & r\lambda & 0 & \cdots & 0 & \cdots \\ \hline 0 & 0 & \cdots & 0 & \mu & 0 & \cdots & 0 & a & r\lambda & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \mu & \cdots & 0 & 0 & a & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \mu & 0 & 0 & \cdots & a & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Fig. 6: **Partitioning of Markov chain and infinitesimal generator matrix**

According to the level partitioning, different submatrices are developed as shown in Equation 2.

$$B_{00} = \begin{bmatrix} -r\lambda & r\lambda & \cdots & 0 \\ 0 & -r\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -r\lambda \end{bmatrix} \quad B_{01} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r\lambda & 0 & \cdots & 0 \end{bmatrix} \quad B_{10} = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r\lambda & 0 & \cdots & \mu \end{bmatrix} \tag{2}$$

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r\lambda & 0 & \cdots & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} -(r\lambda+\mu) & -r\lambda & \cdots & 0 \\ 0 & -(r\lambda+\mu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -(r\lambda+\mu) \end{bmatrix} \quad A_2 = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r\lambda & 0 & \cdots & \mu \end{bmatrix}$$

C. Phase type distribution as a service process

An infinite queue in which arrival process follows the Markovian distribution and service follows a phase type distribution.

Erlang distribution with r stages

Consider an infinite queue with Poisson arrivals with rate λ¸ and the service process follows an Erlang-r distribution with rate $r\mu$ is shown in (**Fig. 7)**.

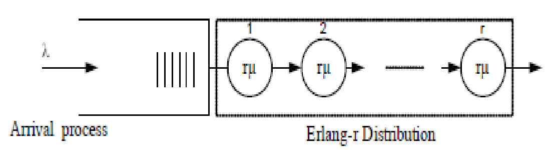
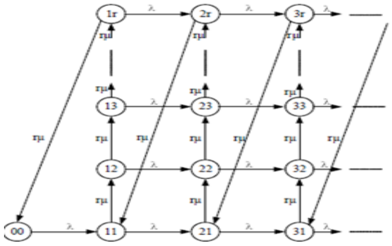


Fig. 7: Queueing model of $M/E_r/1$

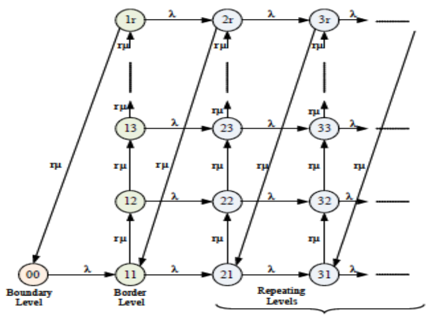
The Markov chain and its infinitesimal generator matrix are shown in (**Fig. 8**).



$$Q_{M/E_r/1} = \begin{array}{c} \\ 00 \\ 11 \\ 12 \\ \vdots \\ 1r \\ 21 \\ 22 \\ \vdots \\ 2r \\ 31 \\ 32 \\ \vdots \\ 3r \\ \vdots \end{array} \begin{array}{ccccccccccccccc} 00 & 11 & 12 & \cdots & 1r & 21 & 22 & \cdots & 2r & 31 & 32 & \cdots & 3r & \cdots \\ -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & a & r\mu & \cdots & 0 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ r\mu & 0 & 0 & \cdots & a & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & a & r\mu & \cdots & 0 & \lambda & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 & 0 & \lambda & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & r\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & a & 0 & 0 & \cdots & \lambda & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a & r\mu & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & r\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Fig. 8: **Markov chain and infinitesimal generator matrix**

Partitioning of the levels in Markov chain and infinitesimal generator matrix is given as in (**Fig. 9).**



$$Q_{M/E_r/1} = \left(\begin{array}{c|cccc|cccc|cccc|c} -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \hline 0 & a & r\mu & \cdots & 0 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ r\mu & 0 & 0 & \cdots & a & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0 & \cdots \\ \hline 0 & 0 & 0 & \cdots & 0 & a & r\mu & \cdots & 0 & \lambda & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 & 0 & \lambda & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & r\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & a & 0 & 0 & \cdots & \lambda & \cdots \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a & r\mu & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & r\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & a & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Fig. 9: Partitioning of Markov chain and infinitesimal generator matrix

The submatrices after partitioning are shown in Equation 3.

$$B_{00} = [-\lambda\ ] \quad B_{01} = [\lambda \quad 0 \quad \cdots \quad 0\ ] \quad B_{10} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ r\lambda \end{bmatrix} \quad A_0 = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (3)$$

$$A_1 = \begin{bmatrix} -(\lambda + r\mu) & -ru & \cdots & 0 \\ 0 & -(\lambda + r\mu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -(\lambda + r\mu) \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r\mu & 0 & \cdots & 0 \end{bmatrix}$$

Hyperexponential distribution with r stages

Consider an infinite queue with Poisson arrivals with rateλ¸ and the service process follows a Hyperexponential distribution with r stages each with rate $\beta_i \mu_i$, $i = 1, 2, \ldots, r$ is shown in (**Fig. 10)**

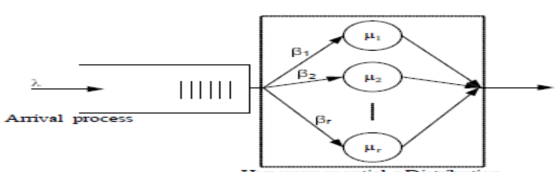
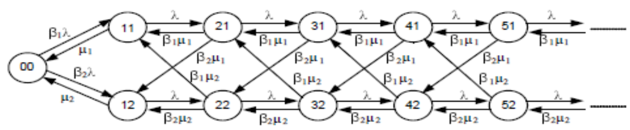


Fig. 10.: **Queueing model of** ***M/Hyper$_r$/1***

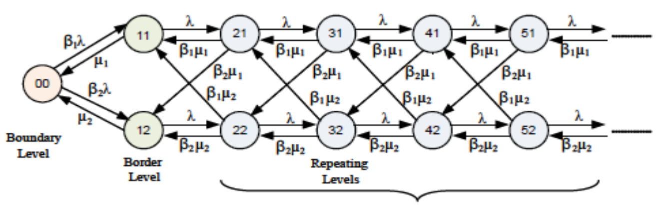
The Markov chain and its infinitesimal generator matrix are shown in (**Fig. 11).**



$$Q_{M/Hyper_2/1} = \begin{array}{c} \\ 00 \\ 11 \\ 12 \\ 21 \\ 22 \\ 31 \\ 32 \\ \vdots \end{array} \begin{array}{cccccccc} 00 & 11 & 12 & 21 & 22 & 31 & 32 & \dots \end{array} \left(\begin{array}{cccccccc} -\lambda & \beta_1\lambda & \beta_2\lambda & 0 & 0 & 0 & 0 & \dots \\ \mu_1 & a_1 & 0 & \lambda & 0 & 0 & 0 & \dots \\ \mu_2 & 0 & a_2 & 0 & \lambda & 0 & 0 & \dots \\ 0 & \beta_1\mu_1 & \beta_2\mu_1 & a_1 & 0 & \lambda & 0 & \dots \\ 0 & \beta_1\mu_2 & \beta_2\mu_2 & 0 & a_2 & 0 & \lambda & \dots \\ 0 & 0 & 0 & \beta_1\mu_1 & \beta_2\mu_1 & a_1 & 0 & \dots \\ 0 & 0 & 0 & \beta_1\mu_2 & \beta_2\mu_2 & 0 & a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Fig. 11: Markov chain and infinitesimal generator matrix

Partitioning of the levels in Markov chain and infinitesimal generator matrix is given as in (**Fig.12**).



$$Q_{M/Hyper_2/1} = \left(\begin{array}{c|cc|cc|cc|c} -\lambda & \beta_1\lambda & \beta_2\lambda & 0 & 0 & 0 & 0 & \dots \\ \hline \mu_1 & a_1 & 0 & \lambda & 0 & 0 & 0 & \dots \\ \mu_2 & 0 & a_2 & 0 & \lambda & 0 & 0 & \dots \\ \hline 0 & \beta_1\mu_1 & \beta_2\mu_1 & a_1 & 0 & \lambda & 0 & \dots \\ 0 & \beta_1\mu_2 & \beta_2\mu_2 & 0 & a_2 & 0 & \lambda & \dots \\ \hline 0 & 0 & 0 & \beta_1\mu_1 & \beta_2\mu_1 & a_1 & 0 & \dots \\ 0 & 0 & 0 & \beta_1\mu_2 & \beta_2\mu_2 & 0 & a_2 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Fig. 12: Partitioning of Markov chain and infinitesimal generator matrix

The resulting submatrices are shown in Equation 4.

$$B_{00} = [-\lambda] \quad B_{01} = [\beta_1\lambda \quad \beta_2\lambda] \quad B_{10} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad A_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad A_1 = \begin{bmatrix} -(\lambda + \mu_1) & 0 \\ 0 & -(\lambda + \mu_2) \end{bmatrix} \tag{4}$$

$$A_2 = \begin{bmatrix} \beta_1\mu_1 & \beta_2\mu_1 \\ \beta_1\mu_2 & \beta_2\mu_2 \end{bmatrix}$$

D. Phase type distribution as an arrival / service process

The queue which using phase-type distributions as an arrival and service process respectively is shown in (**Fig. 13**). In this type of queue the special product operator is used to generate the submatrices of the system without constructing the structured Markov chain and its generator matrix. Kronecker product operator is the special operation which can be perform on the matrices of the phase type distribution used as an arrival and service process. The submatrices of this system can be obtained as shown in Equation 5.

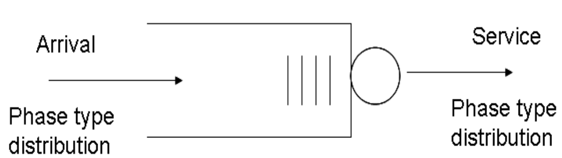


Fig. 13: Queueing model for phase type distribution as an arrival and service

$$B_{00} = T \quad B_{01} = (T^0.\alpha) \otimes \beta \quad B_{10} = I_T \otimes S^0 \quad A_0 = (T^0.\alpha) \otimes I_S \quad A_1 = T \otimes I_S + I_T \otimes S$$

$$A_2 = I_T \otimes (S^0.\beta) \quad B_{s+1} = A_a + A_0 \tag{5}$$

3. RESULTS AND DISCUSSION

The mean flow time of the PH/M/1 queue plotted against the system utilization is given in (**Fig. 14**) for Erlang and hyperexponential distribution.

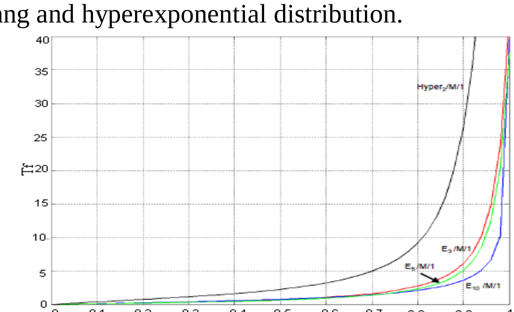


Fig. 14: Mean flow time in the system: PH/M/1 queue: *E*3, *E*5, *E*10, *Hyper*2, *μ* =1

The mean flow time of the M/PH/1 queue plotted against the system utilization is given in (**Fig. 15**). It is observed that the mean flow time of customers in the system with hyperexponential distribution increases faster than the system with Erlang distribution. Here, it is also seen that the number of phases has significant effect on the system mean flow time. An increase in the Erlang phases, mean flow time increases slowly by varying the system utilization and finally increases to infinity when system utilization reaches near to the maximum utilization. Here again, it is seen the same behavior of the system for hyperexponential and Erlang distributions by varying the number of phases of the distribution.

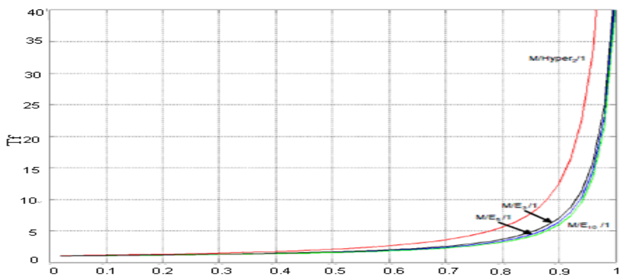


Fig. 15: Mean flow time in the system: M/PH/1 queue: *E*3, *E*5, *E*10 and *H*2

The mean flow time of the PH/PH/1 queue is plotted against the system utilization for different and same phase type distribution as an arrival and service process. (**Fig. 16**) shows the mean flow time of customers in the system of the same phase type distribution against the system utilization. In (**Fig. 17**), the effect of the different phase type distributions as an arrival and service process of the system can be clearly studied

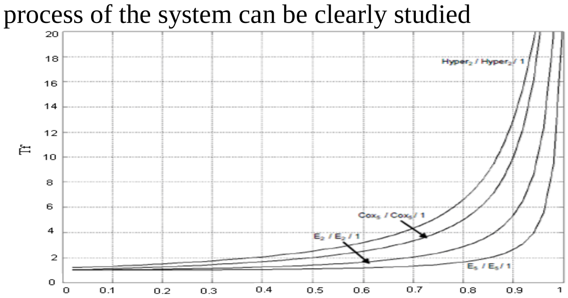


Fig. 16: Mean flow time in the system: PH/PH/1 queue: arrival and service process have same phase-type distributions

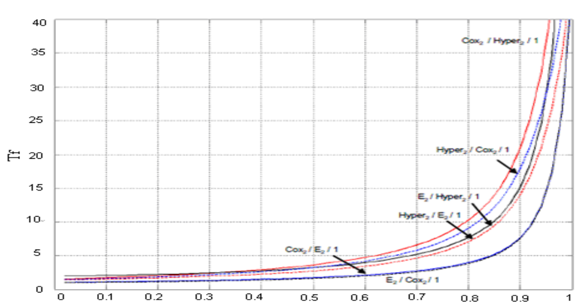


Fig. 17: Mean flow time in the system: PH/PH/1 queue: arrival and service process have different phase-type distributions

4. CONCLUSION

In this paper, we used a vector domain approach to model the quasi birth death systems with vector form phase type distributions. The system is solved using matrix geometric method in which the arrival or service or both processes are modeled through a vector form phase type distribution.

The submatrices of the systems are constructed from structured Markov chain as well as from kronecker product operation. These submatrices are used to trace the flow time of customers in the system. The mean flow time of the different systems having same or different phase type distributions was analyzed.

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