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Lower Bound on Domination Number of Cartesian products of odd cycles

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Abstract: *Let $\gamma(G)$ denote the domination number of a graph G. Let C_n denote the cycle where n is a positive integer with $5 < n$. In this note, a lower bound on the domination number of the Cartesian product of two cycles C_5 and C_n is characterized such that*

$$\gamma(C_5 \mathbin{W} C_n) \geq 3\left\lceil \frac{n-2}{4} \right\rceil + 2\left\lceil \frac{n}{4} \right\rceil$$

Keywords: Cartesian product, domination number, Vizing's conjecture

1. INTRODUCTION

In this paper let $G = (V, E)$ which denotes the vertex set $V(G)$ and the edge set $E(G)$. Instead we write V for vertex set and E for the edge set. The graph G under consideration in this paper is a simple graph, which is a graph without loops and multiple edges. A vertex $v \in V$ is said to dominate if every vertex $V - v$ is neighbor of v. A vertex set $D \subseteq V(G)$ is a ***dominating set*** if every vertex not in D has a neighbor in D. The ***domination number*** $\gamma(G)$ is the minimum size of a dominating set in G. The ***Cartesian product*** of two simple graphs G and H denoted as $G \mathbin{W} H$ is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \times H)$ whenever $x = y$ and $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. For a vertex $x \in V(H)$ set $G_x = G \mathbin{W} \{x\}$ and for a vertex $a \in V(G)$ set $H_a = \{a\} \mathbin{W} H$. The sets G_x and H_a are called the layers of G or H respectively.

Vadim Vizing in 1963 conjectured that the domination number of the Cartesian product of two graphs G and H is always at least as large as the product of the domination numbers of these two graphs; that is, $\gamma(G \mathbin{W} H) \geq \gamma(G)\ \gamma(H)$ which implies the problem of finding the domination number of the graph G.

Upto now, it seems difficult to determine the domination numbers of a particular Cartesian products and only partial results has appeared so far and for details see literature. In this paper, domination number of Cartesian product of two odd cycles C_5 and C_n is studied where n is an odd integers with $5 < n$.

2. MATERIAL AND METHODS

Lower Bound

Throughout this paper, vertices of cycles are indexed from $0, 1, ..., n$ and also the vertices produced by the Cartesian product are indexed from $0, 1, ..., n$. Let $n \times n$ be the grid generated by Cartesian product of two odd cycles C_5 and C_n indexed from $0, 1, ..., n$ with vertex set (x, y). Let D be the domination set containing the vertices of the type (i, j) and (k, l) with their alternate occurrence. Each index i, j, k and l belongs respectively to the sets

$I = \{i \mid i \equiv 0 \pmod 2,\ i \geq 0\}$,
$J = \{j \mid j \equiv 2 \pmod 4,\ j \geq 0\}$,
$K = \{k \mid k \equiv 1 \pmod 2,\ k \geq 0\}$ and
$L = \{l \mid l \equiv 0 \pmod 4,\ l \geq 0\}$ with the respective

bounds on each index as $0 \le i \le 2$, $0 \le j \le n-1$, $0 \le k \le 1$ and $0 \le l \le n-1$.

S. Klavžar and N. Seifter found two results of Cartesian product of two cycles $C_5 \mathrm{WC}_n$ and $C_5 \mathrm{WC}_n$ and found the domination number presented in theorems without proof below. One of the results of S. Klavžar and N. Seifter is presented in a theorem below without proof and depicted as graphs with diamond dots in four figures below for only odd values of n.

Theorem 1. [*S. Klavžar and N. Seifter, p.133*]

$$\gamma(C_5 \mathrm{WC}_n) = \begin{cases} n, & n = 5k \\ n+1, & n \in \{5k+1,\ 5k+2,\ 5k+4\} \end{cases}$$

Furthermore, $\gamma(C_5 \mathrm{WC}_{5k+3}) \le 5(k+1)$.

The values for $n = 5k$, $5k+1$, $5k+2$ and $5k+4$ are depicted for odd values of n where $5 < n$ in the following graphs respectively.

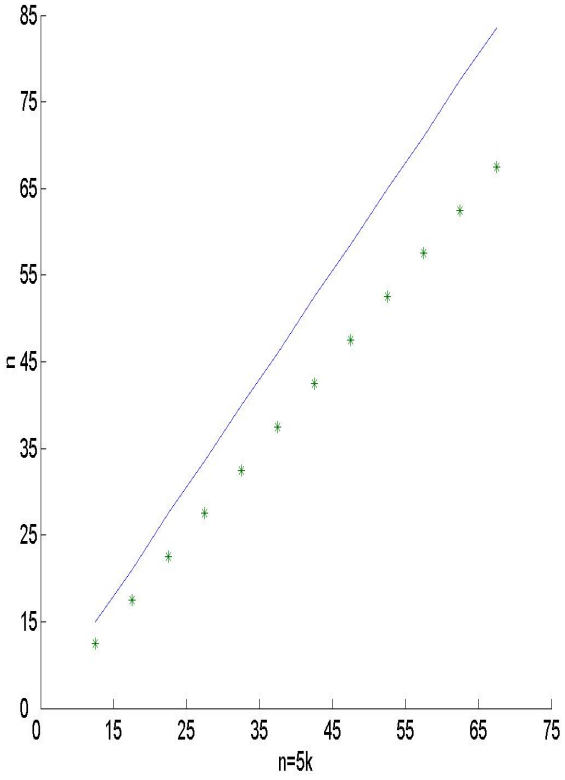


Fig. 1. Calculations for 5k

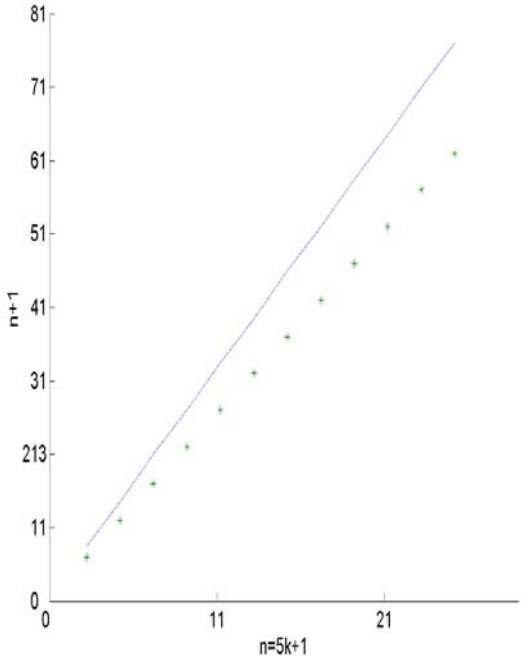


Fig.2. Calculations for 5k+1

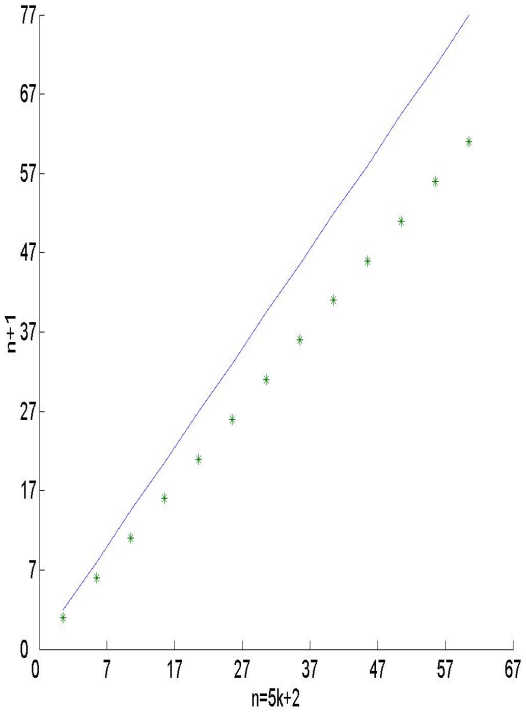


Fig. 3. Calculations for 5k+2

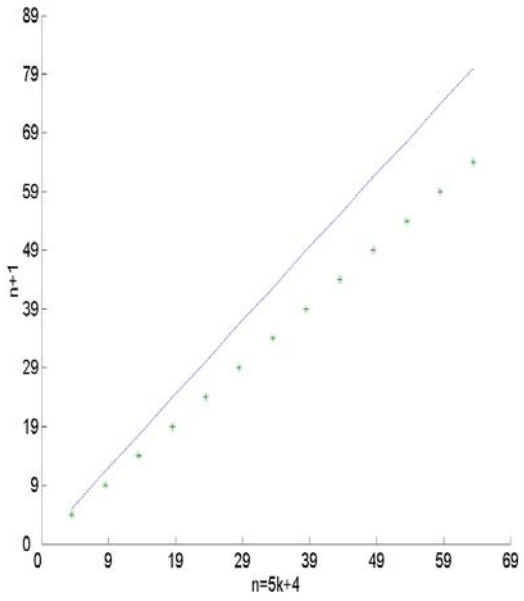


Fig. 4. Calculations for 5k+4

Theorem 2. For an odd n and $5 < n$, the lower bound is $\gamma(C_5 \mathrm{W} C_n) \geq 3\left\lceil \frac{n-2}{4} \right\rceil + 2\left\lceil \frac{n}{4} \right\rceil$

Proof: Let $n \times n$ be the grid generated by the Cartesian product of two cycles C_5 and C_n where n is an odd integer with indexed from the set $\{0,1,...,n\} \times \{0,1,...,n\}$. Let D be the dominating set of the Cartesian product, where the vertex set $v \in D$. Let $v \in D$ and let v be of the form (i, j) where $i \equiv 0 (\mathrm{mod}\ 2)$ and $j \equiv 2 (\mathrm{mod} 4)$ 0. Considering the layers of the product, $i-$layer has $0 \leq i \leq 4$ vertices to be the candidate of being dominating the product and the set is $i \in \{0, 2, 4\}$ as $i \equiv 0 (\mathrm{mod}\ 2)$. The corresponding $j \equiv 2 (\mathrm{mod} 4)$, where the index j runs in the interval $0 \leq j \leq n-1$ which implies $j \in \{2, 6\}$ and the Cartesian product of the sets I and J is $I \times J = \{(i, j) | i \equiv 0 (\mathrm{mod} 2)\ and\ j \equiv 2 (\mathrm{mod} 4)\}$ which implies that each $i-$layer of the product contains $\left\lceil \frac{n-2}{4} \right\rceil$ and there are $|I|\ i-$layers in total and hence we have $3\left\lceil \frac{n-2}{4} \right\rceil$ dominating the $i-$layers.

Similarly, the remaining dominating vertices are of the form (k, l), where, $k \equiv 1 (\mathrm{mod} 2)$ and $l \equiv 0 (\mathrm{mod} 4)$. The $k-$layer of the product has $0 \leq k \leq 3$ vertices belonging to the dominating set D, which are $k \in \{1, 3\}$ and the corresponding l index runs through $0 \leq l \leq n-1$, that is, $l \in \{0,4\}$ and the Cartesian product of the sets K and L is $K \times L = \{(k, l) | k \equiv 1 (\mathrm{mod} 2)\ and\ l \equiv 0 (\mathrm{mod} 4)\}$ which again implies that the $k-$layer of the product contains $\left\lceil \frac{n}{4} \right\rceil$ dominating vertices in each $k-$layer and it occurs $|K| = 2$. Hence we have

$$|I| \left\lceil \frac{n-2}{4} \right\rceil + |K| \left\lceil \frac{n}{4} \right\rceil$$

$$3\left\lceil \frac{n-2}{4} \right\rceil + 2\left\lceil \frac{n}{4} \right\rceil$$

$$\gamma(C_5 \mathrm{W} C_n) \geq 3\left\lceil \frac{n-2}{4} \right\rceil + 2\left\lceil \frac{n}{4} \right\rceil$$

Which proves the lower bounds.

3. CONCLUSION

Results of S. Klavžar and N. Seifter and our results are depicted in four Figs. above. Findings of *S. Klavžar and N. Seifter* are starred and our is a straight line produced by odd n for the same values of as *S. Klavžar and N. Seifter* in which it is an immediate inference that the Cartesian product is bounded below: by $3\left\lceil \frac{n-2}{4} \right\rceil + 2\left\lceil \frac{n}{4} \right\rceil$.

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