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MODIFIED INCOMPLETE BETA FUNCTION RATIO

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Abstract

This paper presents an alternative form of incomplete beta function for m and n by using different values for constructing table of the incomplete beta function for $p = 0.01$ to 1.00. Karl Pearson has edited the tables of incomplete beta function $B_p(m,n)$ and ratio of incomplete beta function to complete beta function.

Keywords: Incomplete beta function; Beta function ratio.

1. Introduction

The Beta function was the first known Scattering amplitude in String theory, first conjectured by Gabriele Veneziano (1942). It also occurs in Preferential Attachment process, a type of Stochastic Urn process. The Beta distribution is the integrand of the Beta function. It can be used to estimate the average time of completing selected tasks in time management problems.

The contributions of Bayes and Richard of the beta probability integral are discussed by Dutka and Hald. Briefly put, in his Essay Bayes obtained an approximation to the two-sided beta probability integral (thus presenting the first use of the incomplete beta function in a probabilistic setting). Price effected improvements, the result being considerably better than the Normal approximation. 'The Bayes-Price results are obtained by approximating the skew beta density by a symmetric beta density.

The wide variation in behavior in different regions of the parameter space, efficient code to evaluate $I(m,n,x)$ involves a number of different subroutines for different parts of this parameter space Didonato and Morris. In Doman is derived a new asymptotic expansion for the incomplete beta function $I(m,n,x)$, which is suitable for large *m*, small *n* and $x > 0.5$.

The incomplete beta function, a generalization of the beta function, denoted $\mathrm{B}_p(m,n)$ .The incomplete beta function coincides with the complete beta function when $p=1$. The relationship between the two functions is like that between the gamma function and its generalization the incomplete gamma function. In Bancroft is derived and discussed recurrence formulae in the Incomplete beta function ratio. The regularized incomplete beta function (or regularized beta function for short) is defined in terms of the incomplete beta function and the complete beta function, denoted $I_p(m,n)$. It is of the importance in probability distribution theory and hence also in obtaining exact values in making of statistical hypotheses. In constructing certain extension of Karl Pearson "Tables of the incomplete Beta-Function Karl Pearson.

2. Incomplete Beta function.

It is well known that incomplete beta function

$$I_p(m,n) = \frac{B_p(m,n)}{B(m,n)} \tag{1}$$

Where $B_p(m,n)$ is defined by the integral in terms of Karl Pearson's notation the incomplete beta function

$$B_p(m,n) = \int_0^x x^{m-1}(1-x)^{n-1}dx \tag{2}$$

and

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m>0, \ n>0 \tag{3}$$

In (2) the upper limit *x* lying between 0 and 1. Evidently the incomplete beta function not only of *m* and *n* but also the upper limit *x*. In the same symbolism the complete beta function $B(m,n)$ should have the subscript 1, it is generally omitted just as subscript ∞ is omitted in writing $\Gamma(n)$ for the complete gamma function. We have complete beta function $B_1(m,n)$, commonly denoted by $B(m,n)$ and is defined by the equation

$$B(m,n) = B_1(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \tag{4}$$

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which is the identical with (2) for $x=1$ and it is of importance in probability distribution theory and also obtaining the exact probability values in making test of statistical hypothesis. In constructing certain extensions karl pearson's "Tables of incomplete beta function" Bancroft .

3. Interchanging the arguments *m* and *n*

By change of variable (3) assume a variety of forms and lead to several interesting relations. If we substitute $x=1-y$, then $dx=-dy$ in (3) we see that the beta function is symmetric in *m* and *n* and we can replace *m* with *n*.

$$B(m,n)=B(n,m) \qquad (5)$$

By the same change of variable

$$B_p(m,n)=\int_{1-x}^{1} y^{n-1}(1-y)^{m-1}dy$$

i,e $B_p(m,n)=B(n,m)-B_{1-p}(n,m)$ (6)

or in another form,

$$B_p(n,m)=B(m,n)-B_{1-p}(m,n) \qquad (7)$$

The incomplete beta function $B_p(n,m)$ an interchange of the two arguments alters the value to $B(m,n)-B_{1-p}(m,n)$. That an interchange of the arguments *m* and *n* has no effect on the complete beta function and at once putting $x=1$ in (6) or (7) and nothing that $B_0(m,n)=B_0(n,m)=0$.

4. Two useful forms of the beta function

(i) If we substitute

$x=z^2/(1+z^2),\ 1-x=1/(1+z^2),\ dx=2z\,dz/(1+z^2)^2$

By introducing this change of variable into

(3) we find that

$$B(m,n)=2\int_0^\infty z^{2m-1}(1+z^2)^{-m-n}dz$$

$$=B(n,m) \qquad (8)$$

In particular, if *m* is replaced by $\frac{1}{2}$ and *n* by $\frac{1}{2}[n-1]$, this relation gives

$$B\left(\frac{1}{2}[n-1],\frac{1}{2}\right)=2\int_0^\infty (1+z^2)^{-\frac{1}{2}}dz$$

$$=\int_{-\infty}^\infty (1+z^2)^{-\frac{1}{2}}dz \qquad (9)$$

(ii) Another useful form is obtained by again substitution $x=\sin^2\theta,\ \ dx=2\sin\theta\cos\theta\ d\theta$ in (3), we have

$$B(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta \qquad (10)$$

In particular,

$$B\left(\frac{1}{2},\frac{1}{2}\right)=2\int_0^{\pi/2}d\theta=\pi \qquad (11)$$

5. Tables of Incomplete beta function

The ratio $B_p(m,n)/B(m,n)$ is the fractional part of the total area under the curve $y=x^{m-1}(1-x)^{n-1}$ between 0 and abscissa $x<1$ and $1-B_p(m,n)/B(m,n)$ or $1-B_p(n,m)/B(n,m)$ is the fractional part remaining between *x* and 1.The tables of $B_p(m,n)$ and $I_p(m,n)$ for $0.01\le p\le 1.00$ and $n\le m\le 50$ was edited by Karl Pearson (1857-1936), British statistician, leading founder of the modern field of statistics, prominent proponent of eugenics, and influential interpreter of the philosophy. The value of *p* starts from 0.01. The values of *m* which are equal or greater than *n* have been given in the tables.

If *m* is less than *n* than the use of this simple identity can be utilized.

$$I_p(m,n)=1-I_{1-p}(n,m)=1-I_{1-p}(m',n') \qquad (12)$$

Where $m'=n$ and $n'=m$ and now $m'>n'$

The $I_p(m,n)$ function has been tabulated to nine decimal places and we can use the results to get the accuracy to more than nine decimal places.

By substitution $x=\sin^2\theta$ in (2), we get

$$B_p(m,n)=2\int_0^{\sin^{-1}\sqrt{p}}\sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta \qquad (13)$$

In particular, for $m=n=3/2,\ 5/2,\ 7/2$ respectively, we get

$$B_p(3/2,3/2)=2\int_0^{\sin^{-1}\sqrt{p}}\sin^2 2\theta\ d\theta$$

$$=\frac{1}{4}\sin^{-1}\sqrt{p}-\frac{1}{16}\sin(4\sin^{-1}\sqrt{p}) \qquad (14)$$

$$B_p(5/2,5/2)=2\int_0^{\sin^{-1}\sqrt{p}}\sin^4\theta\ \cos^4\theta\ d\theta \qquad (15)$$

$$B_p(7/2,7/2) = 2\int_0^{\sin^{-1}\sqrt{p}} \sin^6\theta \cos^6\theta \, d\theta \quad (16)$$

These integrals will be evaluated by using reduction formula.

$$\int \sin^n\theta d\theta = -\frac{\cos\theta \sin^{n-1}\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta d\theta, \; n>0 \quad (17)$$

and

$$\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{2}{3} & \text{when } n \text{ is odd} \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{1}{2}\frac{\pi}{2} & \text{when } n \text{ is even} \end{cases} \quad (18)$$

$\sin^{-1}\sqrt{p}$ have been tabulated by Bliss The tables of. The values of $B_p(3/2,3/2)$ has been used by Hirai in evaluating the moments of order statistics for the Rayleigh Distribution. Three **(Tables 1.2 and 3)** have been constructed to given the values of *m* and *n* for $p = 0.01$ to 1.00 Tables have been constructed to given values of $B_p(3/2,3/2)$, $B_p(5/2,5/2)$ and $B_p(7/2,7/2)$ as shown in the graphs respectively **(Graph 1,2 and 3)**.

6. Conclusion

The different values of $B_p(m,n)$ are used in evaluating the moments of order statistics the Rayleigh distribution. Using an alternative form of incomplete beta function such tables can also be constructed for $B_p(m,n)$ for $m > 7/2$ and $n > 7/2$. These may also be constructed in general form values of *m* and *n*.

Graph–I
The values of Bp(3/2, 3/2) and Ip(3/2, 3/2)

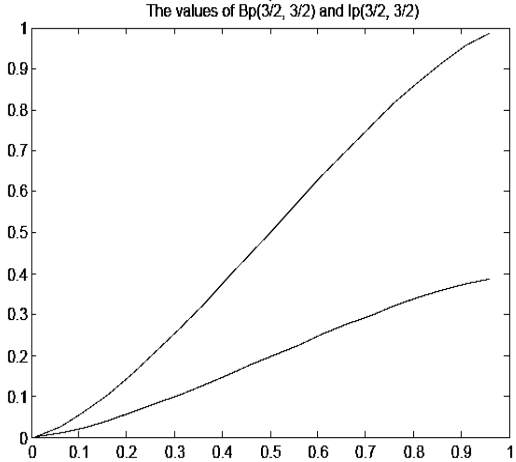

Table-I The values of $B_p(m,n)$ and $I_p(m,n)$ function $m = 1.5$, $n = 1.5$ and $p = 0.01$ to 1.00 $B(m,n) = 0.392699084$

p	$B_p(m,n)$	$I_p(m,n)$
.01	0.000664663	0.001692551
0.05	0.007340738	0.018693037
0.1	0.020437638	0.052044019
0.15	0.036937355	0.094060202
0.2	0.055911902	0.142378489
0.25	0.076773106	0.195501109
0.3	0.099084178	0.252315788
0.35	0.122490239	0.311918832
0.4	0.146684903	0.373530039
0.45	0.171391270	0.436444286
0.5	0.196349541	0.500000000
0.55	0.221307811	0.563555714
0.6	0.246014178	0.626469961
0.65	0.270208843	0.688081168
0.7	0.293614904	0.747684212
0.75	0.315925976	0.804498891
0.8	0.336787179	0.857621510
0.85	0.355761725	0.905939799
0.9	0.372261443	0.947955981
0.95	0.385358343	0.981306963
1.00	0.392699082	1.000000000

Graph–II
The values of Bp(5/2, 5/2) and Ip(5/2, 5/2)

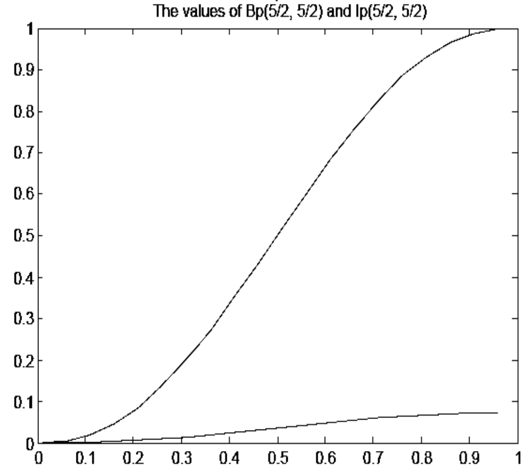

Table-II The values of $B_p(m,n)$ and $I_p(m,n)$ function $m = 5/2$, $n = 5/2$ and $p = 0.01$ to 1.00 $B(5/2, 5/2) = 0.073631078$

p	$B_p(m,n)$	$I_p(m,n)$
0.01	0.000003957	0.000053744
0.05	0.000211745	0.002875758
0.1	0.001132057	0.015374720
0.15	0.002942176	0.039958345
0.2	0.005683482	0.077188625
0.25	0.009320591	0.126584998
0.3	0.013766579	0.186966962
0.35	0.018897773	0.256654846
0.4	0.024564032	0.333609563
0.45	0.030596742	0.415541139
0.5	0.036815539	0.500000000
0.55	0.043034336	0.584458861
0.6	0.049067046	0.666390437
0.65	0.054733305	0.743345154
0.7	0.059864499	0.813033038
0.75	0.064310488	0.873415002
0.8	0.067947596	0.922811375
0.85	0.070688902	0.960041655
0.9	0.072499021	0.984625281
0.95	0.073419333	0.997124242
1.00	0.073631078	1.000000000

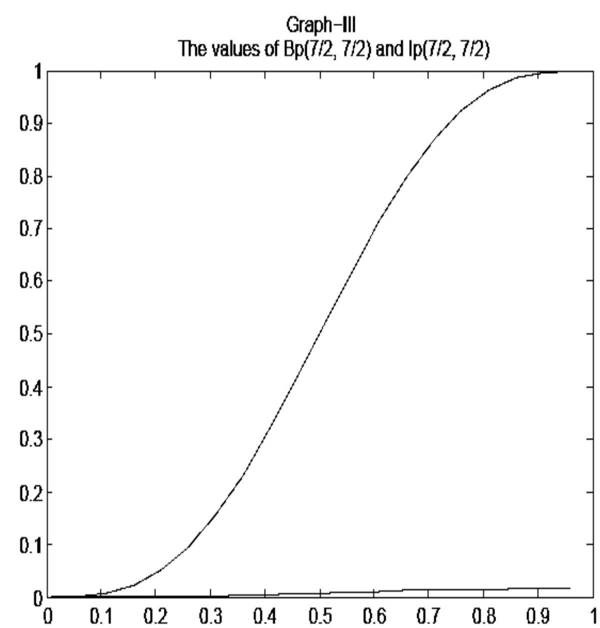

Table-III The values of $B_p(m,n)$ and $I_p(m,n)$ function $m = 7/2$, $n = 7/2$ and $p = 0.01$ to 1.00 $B(7/2, 7/2) = 0.015339802$

p	$B_p(m,n)$	$I_p(m,n)$
0.01	0.000000028	0.000001827
0.05	0.000007233	0.000471531
0.1	0.000073845	0.004813962
0.15	0.000274349	0.017884787
0.2	0.000672059	0.043811414
0.25	0.001307494	0.085235330
0.3	0.002194399	0.143052551
0.35	0.003319882	0.216422663
0.4	0.004647204	0.302950637
0.45	0.006120366	0.398985848
0.5	0.007669904	0.500000000
0.55	0.009219442	0.601014152
0.6	0.010692603	0.697049363
0.65	0.012019926	0.783577337
0.7	0.013145409	0.856947449
0.75	0.014032314	0.914764671
0.8	0.014667749	0.956188585
0.85	0.015065459	0.982115212
0.9	0.015265963	0.995186038
0.95	0.015332575	0.999528469
1.00	0.015339808	1.000000000

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