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IMPROVING THE EFFICIENCY OF HEUN'S METHOD

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Abstract

In this short paper, a modified Heun's method is proposed by replacing the unknown *y* of the interior function. Higher order accuracy is maintained using this new modified version of Heun's Method. This method is used for the solution of Initial Value Problems (IVP). A small modification in the Heun's method has resulted better performance for the computation of numerical solutions. Stability, accuracy and consistency are achieved.

Keywords: Modified Heun's method the solution of Initial Value Problems (IVP).

1. Introduction

Differential Equations arise in many engineering and physical science problems. Recently, these differential equations are being used in other branches of Sciences, *e.g.* Bio-chemistry, bio-technology, Bio-Physics and *etc.* The differential equations arise due to the study of the Physical system. However, many of these differential equations can not be solved using analytical Methods. So, the need of numerical techniques arises to solve these equations (Butcher, 2003).

Numerical techniques for the solution of differential equations are important tools that ever developed with gradual improvement. Various numerical methods have been proposed for the accurate solution of the various types of ordinary differential equations. These methods descritize the system of differential equations in a form map. All maps, obtained using different methods of the same equation must be closely related and have the same goal *i.e.* the dynamics of the differential equations (Jahnke, 2003).

Manually, it is difficult to obtain an accurate solution of the differential equation. However, use of the computer made it is easy for the numerical solution of the differential equations. Thus these methods are now attractive to the researchers and provide an efficient way to get the approximate solutions of the differential equations that were difficult to solve analytically.

Literature review:

Despite many breakthroughs in the history, there is no general method for the solution of every differential equation. Due to this, the differential equations have been classified into different classes. Each particular class of differential equation has further been categorized into subclasses, which can be solved using different methods (Jhanke, 2003).

Jahnke (2003) in his work has commented that there were very little methods during the time of research of differential equations. Jahnke further wrote that solution methods for differential equation became known gradually as many found solution methods but kept them to themselves until their papers were published.

With the span of time, the research into differential equations advances, and with time, its usefulness was realized. However, there was no unique method that could be used to solve every differential equation. So the researches derived different methods for various types of differential equations.

Taylor introduced a method for series of solutions using undetermined coefficients use initial and boundary conditions.

Picard's method was proved efficient algorithm for the solution of differential equation (Palais *et al*., 2009). Despite it unsatisfactory performance, Euler's method came from this method.

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Euler's Method is a type of explicit method that is simple but not practical. When great accuracy is needed, Euler's Method fails (Kreyszig 2006).

We know that the Euler's method compute the slope at point (t_0, y_0). Thus the Euler approximation of $y_1(t)$ can be written as:

$$y_1 = y_0 + \Delta t f(t_0, y(t_0)) \, , \qquad (1)$$

where $\Delta t = t_1 - t_0$; $y(t_0)=y_0$

Thus the Euler Method can be described as (Bultheel and Cools, 2009):

$$y_{n+1} = y_n + \Delta t f(t_n, y_n(t_n)) \qquad (2)$$

where $\Delta t = t_{n+1} - t_n$

Here, we can see that the function $f(t_n, y_n)$ is computed once for the calculation of y_{n+1}, which depends upon y_n and not on the previous values, like $y_{n-1}, y_{n-2}, \cdots$. It is to be noted that the function it self is used instead of $f_2, f_3, \cdots$ that can compute values for $y'', y''', \cdots$ in terms of $f', f'', \cdots$

This method is one of the simplest methods for finding the approximate solution of the differential equation. In this method, the accuracy of the solution depends upon the size of the *h*. Smaller the size of *h*, better the accuracy is achieved. However, with small *h* the large numbers of iterations are required to get the desired accuracy. Thus, this method is cost effective in terms of time.

As discussed earlier that the stability of this method largely depends upon the size of *h*. Butcher (2006) in his work described the Euler method suitable for theoretical study but not for satisfactorily accurate results. In order to achieve high order accuracy and stability in the numerical algorithms, modifications were made which brought new numerical methods.

Runge (1901) noted that Euler's method produces less accurate approximation of the given integral, therefore he proposed for the computation of y_{n+1}, the calculation of inner function at $t_{n+h/2}, y_{n+h/2}$,instead of calculating the inner function at t_n, y_n. By introducing mid-point calculation, eq (1) can be written as mid-point Method as:

$$y_{n+1} = y_n + \Delta t f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2}) \qquad (3)$$

Taking the average of two slopes *i.e.* $f(t_n, y_n), f(t_{n+1}, y_{n+1})$ and doing so changes in equation (2), we obtained Heun's Method ((Heun, 1900), (Ricardo 2009)), which can be described as:

$$y_{n+1} = y_n + 0.5\Delta t \begin{pmatrix} f(t_n, y_n) + \\ f(t_n + \Delta t, y_n + \\ \Delta t f(t_n, y_n)) \end{pmatrix} \qquad (4)$$

The Heun's Method is also known as Predictor-Corrector method. The changes in the Euler's method brought the hierarchy of numerical solution accuracy.

Now, Modified Euler's method can be recovered by inserting forward Euler step in place of unknown values of *y* in equation (3), which can be written as:

$$y_{n+1} = y_n + \Delta t f(t_n + 0.5\Delta t, y_n + 0.5\Delta t f(t_n, y_n)) \qquad (5)$$

Both Modified and Improved Euler's methods are of order two and local error of $O(h^2)$.

High order methods are better for accurate calculations. So, Euler's method becomes the basis for the development of other advanced numerical methods. See for more details ((Budd, 2006), (Heun, 1900), (Hut¡a, 1956), (Kutta, 1901), (Nystrom, 1925), and (Butcher, 2006)).

2. Material and Method

Here, attempt is made to improve the efficiency and accuracy of the existing Heun's method described by eq. (5) making some changes in the existing method.

In order to achieve high order accuracy, the inner function of equation (5) is replaced by the average of the two slopes, we obtain:

$$y_{n+1} = y_n + \Delta t f(t_n + 0.5\Delta t, y_n + K_1) \qquad (6)$$

Where $K_1 = 0.5\Delta t\left(f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right)$ and

$$y^*_{n+1} = y_n + \Delta t f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2})$$

The proposed method is of second order accurate and local error is O (h^3).

3. Results and Discussions

Here, some IVP, given below, are solved using the proposed method and are compared with the Heun's method.

Example 1. $y' = -y; y(0) = 1$;

Exact solution: $y = e^{-t}$

Example 2. $y' = 2y + 3e^t; y(0) = 0$;

Exact Solution: $y = 3(e^{2t} - e^t)$

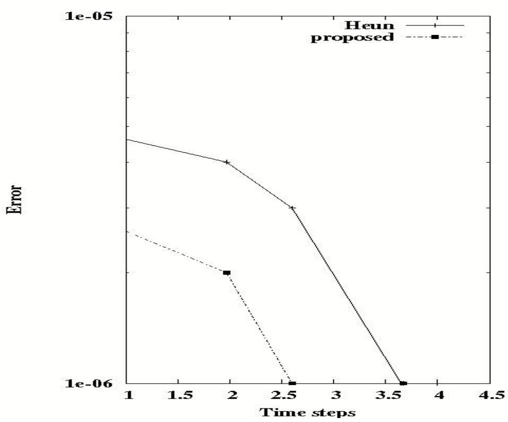

Fig. 1: Error Comparison with Exact solution of example 1: Heun's method Verses Proposed Method

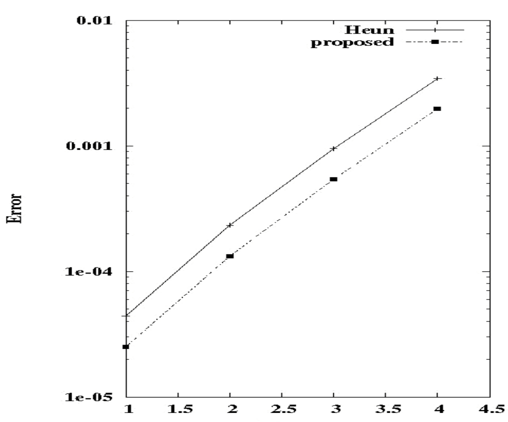

Fig. 2: Error Comparison with Exact solution of example 2: Heun's method Verses Proposed Method

Here, two examples are chosen for the comparison of two methods. The solution so obtained using equation (5) and equation (6) is compared against the exact solution, shown against the differential equations of example 1 and example2.

For numerical computation, time steps is taken as dt = 0.01. The number of iterations performed is 400. The line plots of error are taken at t =1.0, t =2.0, t =3.0, t =4.0

In **(Fig. 1)**, we observe that the solution obtained using proposed method converged at time t=3.0 where as the Heun's method has converged at time t=4.0. This shows that the proposed method performs well in comparison of the Heun's method. Similar sort of performance is observed in figure 2. **(Fig. 2)**, shows the error difference of Proposed and Heun's method with the exact solution of Example 2. The error difference obtained using Proposed method is smaller in comparison of Heun's Method. We can conclude that the proposed method behaves better than the Heun's method.

4. Conclusion

In this paper attention is focused to improve the efficiency of the Heun's method by introducing a change in the existing Heun's method. This is achieved by comparing the solution of linear ODE equation using Heun's method against our proposed modified method. We found that the proposed method works well in comparison of the Heun's method. However, the solution of example 2 has not shown convergence upto t=4.0. Here, the time interval, *dt*, is taken 0.01. However, the error difference is two order less than of the Heun's method. Thus we can say that the proposed changes in the Heun's method has improved its efficiency and forced the solution to converge in fewer time steps, as shown in figure 1.

Future studies will be devoted to apply the proposed method for the solution of non-linear viscoelastic problems (Chandio, *et. al*., 2003 and Chandio *et. al*., 2004). Solution will be obtained for higher order differential equations using smaller time steps, like 0.001,0.002, and 0.005. Results will be compared against Runge-Kutta method of order 4 and Adams-Bashforth method.

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