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An iterative algorithm for the numerical solution of some obstacle problems

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Abstract:

In this paper, we propose an algorithm for solving the obstacle problem. We begin with a suitable initial guess value of the solution, and then construct an iteration algorithm to generate the monotone sequence. We prove the monotone and convergence of the algorithm numerical examples are given for the obstacle problem for a membrane and the elastic-plastic torsion problem.

Keywords: Iteration algorithm, obstacle problems, variational inequalities.

1. Introduction:

Let $\Omega \subset \Re^2$. Consider the obstacle problem: Find $u \in K$ such that

$$E(u) = \inf_{v \in K} E(v), \tag{1.1}$$

where

$$K = \left\{ v \in H_0^1(\Omega) \middle| v \le \psi \ a.e. \text{ in } \Omega \right\}, \tag{1.2}$$

$$E(v) = \int_\Omega \left(\frac{1}{2} k(x) |\nabla u|^2 - fv \right) \tag{1.3}$$

$k(x) \ge k_0 > 0, \nabla v = (\partial v/\partial x_1, \partial v/\partial x_2)^T$ and the obstacle function ψ satisfies the condition $\psi \in H^1(\Omega) \cap C(\Omega)$ and $\psi \ge 0$ on Γ the boundary of the domain Ω. The problems (1.1)-(1.3) describe the equilibrium position u of an elastic membrane constrained to lie above a given obstacle ψ under an external force $f(x) \in L^2(\Omega)$. It is well known that the solution of (1.1) - (1.3) is characterized by the following variational inequality of the first kind: Find $u \in K$ such that

$$\int_\Omega k(x) \nabla u \,.\, \nabla (v-u) dx \ge \int_\Omega f(v-u) dx, \quad \forall v \in K \tag{1.4}$$

Let $V_h \in H_0^1(\Omega)$ be a linear finite element space on the triangulation $\Im_h$. Then the discrete

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$$K_h = \{v_h \in V_h | v_h(x) \le \psi(x), \text{ for any node } x \text{ of } \Im_h\} \quad (1.5)$$

Then the approximation of problem (1.1)-(1.3) is to find $u_h \in K_h$ such that

$$E(u_h) = \inf_{v_h \in K_h} E(v_h) \quad (1.6)$$

It is equivalent to the variational inequality of the first kind: Find $u_h \in K_h$ *such* that

$$\int_\Omega k(x)\nabla u_h \cdot \nabla(v_h - u_h)dx \ge \int_\Omega f(v_h - u_h)dx, \quad \forall v_h \in K_h \quad (1.7)$$

The error estimate in H^1 norm or L^∞ norm was proved in [3,14,15]. Since the discrete maximum principle is only hold for the linear elements, here we use the linear finite elements space V_h *h*and avoid the higher order finite elements. The discrete maximum principle plays a role rule in the analysis of our algorithm.

The obstacle problem is one of the simplest unilateral problems; it arises when modeling a constrained membrane in the classical elasticity theory. It is the typical example of the elliptic variational inequality of the first kind. Many important problems such as the torsion of an elastic–plastic cylinder, the Stefan problem can be formulated by transformation to an obstacle problem. Several comprehensive monographs can be consulted for the theory and numerical solution of variational inequalities, e.g. [1, 2, 3, 4, 5, 6]. Since obstacle problems are highly nonlinear, it is difficult for the computation of approximate solutions. The approximate solution of obstacle problems is usually solved by variable projection method, for example, the relaxation method [3], multi level projection method [7], multi grid method [8, 9, 10, 11] and projection method [12, 13] for nonlinear complementarity problems. Recently the authors in [16, 17] analyze two iteration algorithms, they need to solve linear systems in each iteration step in their algorithms.

In this paper, we will propose a simple effect algorithm to solve the obstacle problem (1.7) by iteration. We first solve a suitable linear system as the initial guess value of our iteration algorithm, then we use the similar relaxation method of Glowinski [3] and obtain amonotonic sequence to approximate the solution. In the next section we present the algorithm and prove the convergence. We report some numerical experiments in section 3.

2. The iterative algorithm:

Denote the $\phi_1(x), \phi_2(x), \ldots, \phi_N(x)$ the basis functions at nodes $x_1, x_2, \ldots, x_N$ of linear elements space V_h and A_h be the $N \times N$ matrix with entries $a_{i,j} = \int_\Omega k(x)\nabla\phi_i \cdot \nabla\phi_j \, dx$. The load vector **f** is $f_i = \int_\Omega f\phi_i dx$. As we use the linear elements space V_h, then the matrix A_h is a symmetric positive definite matrix with the positive diagonal entries $a_{i,i} > 0$, non-positive off-diagonal entries $a_{i,j} \le 0$ $(i \ne j)$ and is irreducibly diagonally dominant. By the discrete maximum principle, we see that the matrix is the monotonic about the vector $y = (y_1, \ldots, y_N)^T$ and $z = (z_1, \ldots, z_N)^T$

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$$(A_h y)_i \le (A_h y)_i, \quad i=1,2,\ldots,N. \Rightarrow y_i \le z_i,\ i=1,2,\ldots,N. \tag{2.1}$$

Denote $(\psi_i)=\psi(x_i)$ for $i=1,2,\ldots,N$ and $\Psi=(\psi_1,\ldots,\psi_N)$.

Then the problem (1.6) is the following forms,

$$J(\mathrm{u})=\min_{v\in K} J(\mathrm{v})()$$

$$J(\mathrm{v})=\frac{1}{2}(A_h \mathrm{v}, v)-(\mathrm{f}, \mathrm{v}),$$

$$K=\left\{v\in \Re^N \,\middle|\, v_i \le \psi_i,\ i=1,2,\ldots,N\right\}$$

From [16], we can rewrite the above problem in to the complementary problem.

Lemma 2.1 [16]: *Let* $\mathrm{u}=(u_1,\ldots,u_N)\in \Re^N$ *and* $u_h(x)=\sum_{i=1}^{N} u_i\phi_i(x)$, *and then* u_h *is a solution to (1.7) if and only if* u *is a solution to the following complementary problem*□

$$\begin{cases} u_i \le \psi_i, \\ (A_h \mathrm{u})_i \le f_i, \\ \left((A_h \mathrm{u})_i - f_i\right)(\psi_i - u_i)=0 \end{cases} \tag{2.2}$$

For all $i=1,2,\ldots,N$

Basing the Lemma 2.1, we propose the following iteration algorithm for solving the problem (2.2).

Algorithm 2.2: (i) *Given the initial guess value* $\mathrm{u}^{(0)}$ *satisfies the inequalities*

$$u_i^{(0)} \le \psi_i, \quad \left(A_h \mathrm{u}^{(0)}\right)_i \le f_i, \quad i=1,2,\ldots,N \tag{2.3}$$

One of the simple way is to solve the linear equation.

$$\left(A_h \mathrm{u}^{(0)}\right)_i = \min\left(f_i, (A_h\Psi)_i\right), \quad i=1,2,\ldots,N \tag{2.4}$$

(ii) *For* $k=0,1,2,\ldots,$ *we let*

$$H_1^{(k)}=\left\{i \,\middle|\, \left(A_h \mathrm{u}^{(k)}\right)_i = f_i\right\}$$

$$H_2^{(k)}=\left\{i \,\middle|\, \left(A_h \mathrm{u}^{(k)}\right)_i = f_i\right\}$$

Then we update the $\mathrm{u}^{(k+1)}$ by

$$u_i^{(k+1)}=\begin{cases} u_i^{(k)}, & \text{if } i\in H_1^{(k)} \\ \left(1-\omega_i^{(k)}\right)u_i^{(k)}+\omega_i^{(k)}\psi_i, & \text{if } i\in H_2^{(k)} \end{cases} \tag{2.5}$$

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Where the parameter $0<\omega_i^{(k)}\le$

$$\omega_i^{(k)} = \begin{cases} 1, & \text{if } u_i^{(k)} = \psi_i \\ \min\left(1, \dfrac{f_i - \left(A\mathrm{u}^{(k)}\right)_i}{a_{i,i}\left(\psi_i - u_i^{(k)}\right)} - \omega_i^{(k)}\right), & \text{if } u_i^{(k)} < \psi_i \end{cases} \tag{2.6}$$

Theorem 2.3: *The iteration algorithm 2.2 is well defined.*

Proof. By induction, if we assume that

$$u_i^{(k)} \le \psi_i, \qquad \left(A_h \mathrm{u}^{(k)}\right)_i \le f_i, \quad i = 1, 2, \ldots, N. \tag{2.7}$$

We only need to prove that by (2.5)-(2.6)

$$u_i^{(k+1)} \le \psi_i, \qquad \left(A_h \mathrm{u}^{(k+1)}\right)_i \le f_i, \quad i = 1, 2, \ldots, N \tag{2.8}$$

As $0 < \omega_i^{(k)} \le 1$, then from the first hypothesis of (2.7) we have $u_i^{(k+1)} = u_i^{(k)} \le \psi_i$ for $i \in H_1^{(k)}$ and $u_i^{(k+1)} = \left(1 - \omega_i^{(k)}\right) u_i^{(k)} + \omega_i^{(k)} \psi_i \le \psi_i$ for $i \in H_2^{(k)}$. That is $u_i^{(k+1)} \le \psi_i$ for all $i = 1, 2, \ldots, N$.

It remains to explain the second inequality in (2.8).

For any $i \in H_1^{(k)}$, we have

$$\begin{aligned} \left(A_h \mathrm{u}^{(k+1)}\right)_i &= \sum_{j=1}^{N} a_{i,j} u_j^{(k+1)} = \sum_{j \in H_1^{(k)}} a_{i,j} u_j^{(k+1)} + \sum_{j \in H_2^{(k)}} a_{i,j} u_j^{(k+1)} \\ &= \sum_{j \in H_1^{(k)}} a_{i,j} u_j^{(k)} + \sum_{j \in H_2^{(k)}} a_{i,j} \left(\left(1 - \omega_j\right) u_j^{(k)} + \omega_j \psi_j\right) \\ &= \sum_{j=1} a_{i,j} u_j^{(k)} + \sum_{j \in H_2^{(k)}} a_{i,j} \omega_j \left(\psi_j - u_j^{(k)}\right) \end{aligned} \tag{2.9}$$

By the property of matrix A_h, we have $a_{i,j} \le 0 \ (i \ne j)$. So the last term in (2.9) is non-positive.

Thus

$$\left(A_h \mathrm{u}^{(k+1)}\right)_i \le \sum_{j=1}^{N} a_{i,j} u_j^{(k)} = \left(A_h \mathrm{u}^{(k)}\right)_i \le f_i, \qquad i \in H_1^{(k)} \tag{2.10}$$

For any $i \in H_1^{(k)}$, again we have

$$\begin{aligned} \left(A_h \mathrm{u}^{(k+1)}\right)_i &= \sum_{j=1}^{N} a_{i,j} u_j^{(k+1)} = \sum_{j \in H_1^{(k)}} a_{i,j} u_j^{(k+1)} + \sum_{j \in H_2^{(k)}} a_{i,j} u_j^{(k+1)} \\ &= \sum_{j \in H_1^{(k)}} a_{i,j} u_j^{(k)} + \sum_{j \in H_2^{(k)}} a_{i,j} \left(\left(1 - \omega_j^{(k)}\right) u_j^{(k)} + \omega_j^{k} \psi_j\right) \end{aligned}$$

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$$= \sum_{j=1} a_{i,j} u_j^{(k)} + \sum_{j \in H_2^{(k)}} a_{i,j} \omega_j^k \left(\psi_j - u_j^{(k)} \right) \qquad (2.11)$$

$$= \left(A_h \mathrm{u}^k \right)_i + \sum_{j \in H_2^{(k)}} \omega_j^{(k)} a_{i,j} \left(\psi_j - u_j^{(k)} \right)$$

$$\leq \left(A_h \mathrm{u}^k \right)_i + \omega_j^{(k)} a_{i,j} \left(\psi_j - u_j^{(k)} \right)$$

$$\leq f_i$$

Here the last inequality we have applied the condition (2.6). Thus

$$\left(A_h \mathrm{u}^{(k+1)} \right)_i \leq f_i, \qquad i = 1, 2, \ldots, N \qquad (2.12)$$

It completes the proof.

Theorem 2.4: *Let* $\mathrm{u}^{(k)}$ $(k = 1, 2, \ldots)$ *be the sequence generated by Algorithm 2.2, and let u be the solution of the non linear equations (2.2). Then* $\mathrm{u}^{(k)}$ *converge to u as k tends to* ∞,

$$\lim_{k \to \infty} u_i^{(k)} = u_i, \qquad i = 1, 2, \ldots, N. \qquad (2.13)$$

Proof. In the proof of theorem 2.3, we have seen that for $i \in H_1^{(k)}$,

$$u_i^{(k+1)} = u_i^{(k)}$$

and for $i \in H_2^{(k)}$

$$u_i^{(k+1)} = \left(1 - \omega_i^{(k)}\right) u_i^{(k)} + \omega_i^{(k)} \psi_i \geq \left(1 - \omega_i^{(k)}\right) u_i^{(k)} + \omega_i^{(k)} u_i^{(k)} \geq u_i^{(k)}$$

So for every *i*, the sequences $u_i^{(k)}$ $(k = 1, 2, \ldots)$ increase and have the upper bound ψ_i.

Then the limit, $\lim_{k \to \infty} u_i^{(k)}$ exists. As $u_i^{(k)} \leq \psi_i$ and $\left(A_h \mathrm{u}^{(k)} \right)_i \leq f_i$, , we have

$$\lim_{k \to \infty} u_i^{(k)} \leq \psi_i, \qquad \lim_{k \to \infty} \left(A_h \mathrm{u}^{(k)} \right)_i \leq f_i$$

By the relation (2.5), we can see that the limit $\lim_{k \to \infty} u_i^{(k)}$ also satisfies the third equation of (2.2).

Thus $\lim_{k \to \infty} u_i^{(k)} = u_i$ .We complete the proof.

Remark2.5: The algorithm 2.2 is a very easy implement scheme. It is parallel in the computation of each $u_i^{(k+1)}$ in (2.5)-(2.6). If we compute the $u_i^{(k+1)}$ consequently from $i = 1, 2, \ldots, N$, we actually can use better parameter $0 < \omega_i^{(k)} \leq 1$ to replace the one in (2.6) by

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$$\omega_i^{(k)} = \begin{cases} 1, & \text{if } u_i^{(k)} = \psi_i \\ \min\left(1, \dfrac{f_i - \left(Au^{(k)}\right)_i - \sum_{j<i, j\in H_2^{(k)}} a_{i,j}\omega_j^{(k)}\left(\psi_j - u_j^{(k)}\right)}{a_{i,j}\left(\psi_i - u_i^{(k)}\right)}\right), & \text{if } u_i^{(k)} < \psi_i \end{cases} \quad (2.14)$$

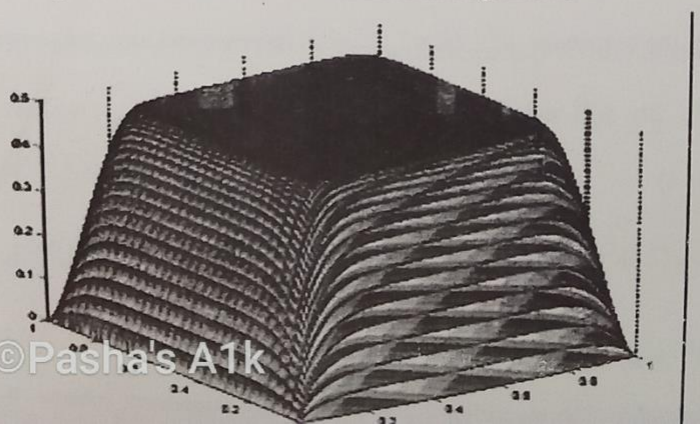
Remark 2.6: The algorithm is similar to the algorithm of the classical relaxation method of Glowinski [3] with the same initial guess values. We begin with the suitable initial value and generate the monotonic sequence. The algorithm of Glowinski [3] begins with arbitrary initial guess value, and then they are not getting the monotonic sequence. So if we can find more suitable initial guess value $u^{(0)}$ than (2.4), then the convergence will be faster. One possible way is to use the solution in coarse mesh as the initial guess value as that in [7].

3. Numerical results:

In this section, some numerical examples are given for the obstacle problem for membrane and the elastic-plastic torsion problem.

Example 3.1: In this example, we let the domain $\Omega=(0,1)\times(0,1)$, $k(x,y)=1$, the force f is 50 and the obstacle is the plane $\psi(x,y)=0.5$. The solution of surface in the grid level $h=1/128$ is plotted in **Fig. 3.1,** which shows an interior flat region where the solution reaches the obstacle $\psi(x,y)=0.5$. We use the Algorithm 2.2 with Remark 2.5 and see that our algorithm performs very effectively in practice when the mesh is refined. The initial guess value $u^{(0)}$ is given by solving equation (2.4). The iteration number is presented in **Table 3.1** for various meshes by the stop criteria $\max\left|u_i^{(k+1)} - u_i^{(k)}\right| \le 10^{-16}$. The iteration number seems not small but the operation in the iteration is very simple.

Fig. 3. 1. The solution surface of an obstacle problem.



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Table 3.1. The Iterative Numbers of Algorithm 2.2

h	1/8	1/16	1/32	1/64	1/128
Iteration numbers	3	24	83	288	1019

Example 3.2: We consider the torsion of an elastic-plastic cylinder of cross-section

$$\Omega=(0,1)\times(0,1),\ [1,3]$$

$$\int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq C\int_\Omega (v-u)\,dx, \qquad \forall v\in K,\quad u\in K \tag{3.1}$$

where

$$K=\left\{v\in H_0^1(\Omega)\,\middle|\, |\nabla v(x)|\leq 1 \ \ a.e.\ in\ \Omega\right\} \tag{3.2}$$

The problem (3.1) and (3.2) can be shown to be equivalent to

$$E(v)=\frac{1}{2}\int_\Omega |\nabla v|^2\,dx + C\int_\Omega v\,dx \ \rightarrow \min \tag{3.3}$$

subject to

$$\left\{v\in H_0^1(\Omega)\,\middle|\, v\leq \psi \ \ a.e.\ in\ \ \Omega\right\} \tag{3.4}$$

If we additionally choose the constant twist angle $C<0$, then (3.3) form an obstacle problem (1.1) with the obstacle function,

$$\psi(x,y)=dist(x,\ \partial\Omega)=\min_{y\in\partial\Omega}\|x-y\| \qquad \forall\, x\in\Omega$$

For $\Omega=(0,1)\times(0,1)$ and $C=-5,-10,-20$ we used our algorithm to obtain an approximated solution to the problem (3.3) and (3.4), with $N=64$. In **Fig. 3.2~ 3.3, 3.4,** and **Table 3.2** we show the plastic region $|\nabla u|=1$ filled with * while its complement characterizes the elastic region in white.

Fig. 3.2: The elastic-plastics problem with C = -10

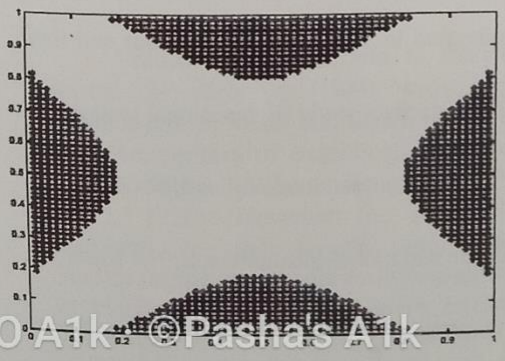
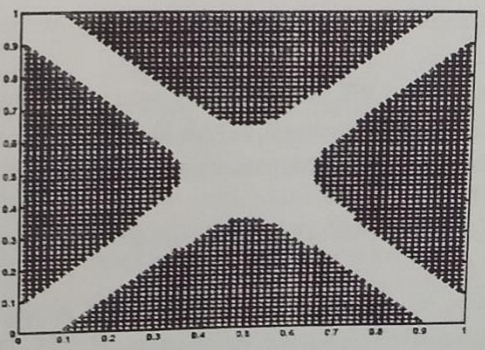


Fig. 3.3: The elastic-plastics problem with C = - 5



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Fig. 3.4. The elastic-plastics problem with C = - 20

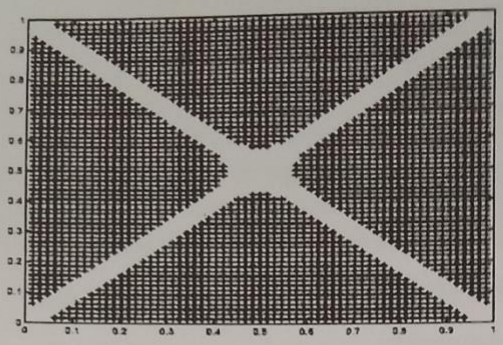


Table 3.2: The Iterative Numbers of Algorithm 2.2 for the Example 3.2 with $C = -5$

h	1/8	1/16	1/32	1/64	1/128
iteration number	44	134	488	1720	5863

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