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On Exact Solutions of the Equations of Motion of an Incompressible Fluid of Variable Viscosity in the Presence of Unknown External Force

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Abstract:

Some exact solutions of equations describing the motion of an incompressible fluid of variable viscosity in the presence of unknown force for an arbitrary state equation with no heat addition are determined in streamlined coordinate system (Φ, Ψ). The solutions involve arbitrary function(s) enabling to construct a large number of solutions to the flow equations.

Keywords: Steady plane flows, Martin's variables, incompressible fluid, variable viscosity, thermal conductivity and Navier-Stokes equations.

1. Introduction

Martin (1971) developed an elegant method for the study of incompressible viscous fluid flows. He introduced curvilinear coordinates (Φ, Ψ) in the plane of flow, in which the coordinate lines Ψ equal to constants are the streamlines of the flow and the coordinate lines Φ equal to constant are left arbitrary. He discussed the geometry of some flows and suggested fifteen different types of flows which could be studied using the new approach. Martin's method was extended by Naeem (1984) to steady-state, plane, constant viscosity, compressible Navier-Stokes equations, and determined some exact solutions for parallel, radial and circular flows. Barron (1989), Barron and Naeem (1989), Naeem and Barron (1990), Barron and Naeem (1991) further extended the Martin's method to study numerically the fluid flow equations and the flows past bodies for inviscid fluids. In their studies they took $\Phi = x$. Naeem and Nadeem (1996), following Martins, transformed the equations governing the steady plane flow of an incompressible fluid of variable viscosity in the absence of external force into a new system of equations in (Φ, Ψ) - system. Naeem and Nadeem determined, by making (Φ, Ψ)-system orthogonal, some exact solutions of the flow equations indicating applicability of some solutions to the physically possible situations. Naeem and Nadeem also deduced some relations between viscosity and temperature which were obtained by Prandtl (1928) and Poiseuille (1860) employing experimental methods. They also indicated that a large number of relations between viscosity and temperature can be constructed.

The objective of the present work is to determine some exact solutions of the equations of motion of an incompressible fluid of variable viscosity in the presence of unknown external force for arbitrary state equation. We mention that no work on the exact solutions of the equations of motion of an incompressible fluid of variable viscosity in the presence of unknown external force for arbitrary state equation has been reported in the literature at yet. To achieve our objective, following Naeem and Nadeem (1996), we transform flow equations into a new system of equations in (Φ, Ψ) system with

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speed, viscosity, vorticity, generalized energy, temperature, the transformation matrices and the components of the external force as the unknown functions. By placing certain conditions on the transformations matrices, a priori, exact solutions are determined for vortex, radial and parallel flows. In each flow the differential equation which the viscosity and unknown external force must satisfy is determined using integrability condition for the generalized energy function L. The expressions for the unknown external force are determined for all flows. The exact solutions involve arbitrary functions and this arbitrariness of functions allows us to construct a large number of exact solutions to the flow equations.

Basic flow equations

The basic non-dimensional equations governing the steady plane flow of an incompressible fluid of variable viscosity in the presence of an unknown external force with no heat addition are:

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = -p_x + \frac{1}{Re}(2\mu u_x)_x + \frac{1}{Re}\{\mu(u_y + v_x)\}_y + \lambda^* f_1 \tag{2}$$

$$uv_x + vv_y = -p_y + \frac{1}{Re}(2\mu v_y)_y + \frac{1}{Re}\{\mu(u_y + v_x)\}_x + \lambda^* f_2 \tag{3}$$

$$uT_x + vT_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\{2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2\} \tag{4}$$

$$\mu = \mu(T) \tag{5}$$

where u, v are the velocity components, p the pressure, T the temperature, μ the viscosity, Re the Reynolds number, Pr the Prandtl number, Ec the Eckert number, ρ the density of the fluid and f_1, f_2 are the components of the external force. In equations (2) and (3), λ^* is a non-dimensional number, and in case of motion under the gravitational force, λ^*, is called the froude number (Fr).
We define the following functions:

$$\omega = v_x - u_y$$

$$L = p + \frac{(u^2 + v^2)}{2} - \frac{2\mu u_x}{Re} \tag{6}$$

In term of these functions, the system of equations (1 – 5) is replaced by the following system:

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$$u_x + v_y = 0 \tag{7}$$

$$-v\omega = -L_x + \frac{1}{Re}\{\mu(u_y + v_x)\}_y + F_1 \tag{8}$$

$$u\omega = -L_y - \frac{4(\mu u_x)_y}{Re} + \frac{1}{Re}\{\mu(u_y + v_x)\}_x + F_2 \tag{9}$$

$$\omega = v_x - u_y \tag{10}$$

$$uT_x + vT_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\{2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2\} \tag{11}$$

$$\mu = \mu(T) \tag{12}$$

of six equations in eight unknowns u, v, L, μ, T, ω, F_1, F_2 as functions of x, y. The advantage of this system over the original system is that the order of the partial differential equations (2) and (3) has decreased from two to one. In equation (8) and (9) for convenience we have put $F_1 = \lambda^* f_1$, $F_2 = \lambda^* f_2$.

Transformation of basic flow equations in the streamlined coordinate system (φ, ψ)

If the arbitrary curves Φ(x, y) = constant and the streamlines Ψ(x, y) = constant, generate a curvilinear net in the physical frame, the system of equations (7–12), following Naeem and Nadeem (1996), is transformed to the following system:

$$q = \frac{\sqrt{E}}{W} \tag{13}$$

$$\begin{aligned} J\omega = {} & -J L_\varphi + A_\varphi\left\{(F^2 - J^2)\frac{\sin 2\alpha}{2E} + \frac{FJ\cos 2\alpha}{E}\right\} \\ & - A_\psi\{F\sin\alpha\cos\alpha + J\cos^2\alpha\} \\ & + B_\varphi\left\{-\frac{F^2\cos 2\alpha}{E} + \frac{2FJ\sin 2\alpha}{E} + \frac{J^2\cos 2\alpha}{E}\right\} \\ & + B_\psi\{F\cos 2\alpha - J\sin 2\alpha\} \\ & + \frac{J}{\sqrt{E}}\{F(F_1\cos\alpha + F_2\sin\alpha) + J(F_1\cos\alpha + F_2\sin\alpha)\} \end{aligned} \tag{14}$$

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$$0 = -JL_\phi + A_\phi\{F\sin\alpha\cos\alpha - J\sin^2\alpha\} - A_\psi\{E\sin\alpha\cos\alpha\} + B_\phi\{J\sin 2\alpha - F\cos 2\alpha\} + B_\psi\{E\cos 2\alpha\} + J\sqrt{E}\,(F_1\cos\alpha + F_2\sin\alpha) \tag{15}$$

$$\frac{\left(\frac{GT_\phi}{J} - \frac{FT_\psi}{J}\right)_\phi + \left(\frac{ET_\psi}{J} - \frac{FT_\phi}{J}\right)_\psi}{J\,Re\,Pr} = -\frac{Ec\,Re\left(A^2 + 4B^2\right)}{4\mu} + \frac{qT_\phi}{\sqrt{E}} \tag{16}$$

$$\omega = \frac{\left(\frac{F}{W}\right)_\phi - \left(\frac{E}{W}\right)_\psi}{W} \tag{17}$$

$$K = \frac{\left(\frac{W\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\Gamma_{12}^2}{E}\right)_\phi}{W} = 0 \tag{18}$$

$$\mu = \mu(T) \tag{19}$$

which in Φ and Ψ are considered as independent variables. This is a system of seven equations in ten unknown functions E, F, G, W, L, T, q, μ, F_1, F_2. In equations (14 - 16), the functions A and B are given by

$$A = \frac{4\mu}{J\,Re}\left[\left(\frac{F\sin\alpha}{\sqrt{E}} + \frac{J\cos\alpha}{\sqrt{E}}\right)(q_\phi\cos\alpha - q\sin\alpha\,\alpha_\phi) - \sqrt{E}\sin\alpha\,(q_\psi\cos\alpha - q\sin\alpha\,\alpha_\psi)\right] \tag{20}$$

$$B = \frac{\mu}{J\,Re}\left[q_\phi\left(-\frac{F\cos 2\alpha}{\sqrt{E}} + \frac{J\sin 2\alpha}{\sqrt{E}}\right) + q_\psi\sqrt{E}\cos 2\alpha + q\left(\frac{F\sin 2\alpha}{\sqrt{E}} + \frac{J\cos 2\alpha}{\sqrt{E}}\right)\alpha_\phi - q\sqrt{E}\sin 2\alpha\,\alpha_\psi\right] \tag{21}$$

The objective of the present work is to seek solutions of the equations (13 – 19) by placing a prior condition on the matrices E, G for an arbitrary state equation (19) when the system (Φ, Ψ) is orthogonal. When the system(Φ, Ψ) is orthogonal,

$$F = 0 \tag{22}$$

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Transformation of the flow equation in streamlined coordinate system (φ, ψ) into the natural orthogonal net – (ζ, η)

If a natural (ζ, η) coordinate system is chosen, where η = constant are streamlines and ζ = constant the steady plane flow of an incompressible fluid of variable viscosity in the presence of external force takes the form:

$$q = \frac{\Psi'}{g_2} \tag{23}$$

$$0 = -L_\zeta + g_1(F_1\cos\alpha + F_2\sin\alpha) + B_\zeta\sin 2\alpha + B_\eta\frac{g_1\cos 2\alpha}{g_2} - A_\zeta\sin^2\alpha - A_\eta\frac{g_1\sin\alpha\cos\alpha}{g_2} \tag{24}$$

$$\omega = -\frac{L_\eta}{\Psi'} + \frac{g_2F_2\cos\alpha}{\Psi'} - \frac{g_2F_1\sin\alpha}{\Psi'} + B_\zeta\frac{g_2\cos 2\alpha}{g_1\Psi'} - B_\eta\frac{\sin 2\alpha}{\Psi'} - A_\zeta\frac{g_2\sin 2\alpha}{2\Psi' g_1} - A_\eta\frac{\cos^2\alpha}{\Psi'} \tag{25}$$

$$\left(\frac{g_2T_\zeta}{g_1}\right)_\zeta + \left(\frac{g_1T_\eta}{g_2}\right)_\eta = -\frac{Ec\,Pr\,Re^2\,g_1g_2}{4\mu}\left(A^2 + 4B^2\right) + Re\,Pr\,\Psi'T_\zeta \tag{26}$$

$$\omega = -\frac{\left(\frac{g_1\Psi'}{g_2}\right)_\eta}{g_1g_2} \tag{27}$$

$$\left(\frac{g_{2\zeta}}{g_1}\right)_\zeta + \left(\frac{g_{1\eta}}{g_2}\right)_\eta = 0 \tag{28}$$

where

$$A = 4v\left[\frac{\cos\alpha}{g_1}\left(q_\zeta\cos\alpha - \frac{\Psi'\sin\alpha}{g_2}\alpha_\zeta\right) - \frac{\Psi'\sin\alpha}{g_2}\left(\frac{q_\eta\cos\alpha}{\Psi'} - \frac{\sin\alpha}{g_2}\alpha_\eta\right)\right] \tag{29}$$

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$$B = \upsilon\left[\frac{1}{g_1}\left(\frac{\Psi'}{g_2}\right)_\zeta \sin 2\alpha + \frac{\Psi' \cos 2\alpha}{g_1 g_2}\alpha_\zeta + \frac{1}{g_2}\left(\frac{\Psi'}{g_2}\right)_\eta \cos 2\alpha - \frac{\Psi' \sin 2\alpha}{g_2^2}\alpha_\eta\right] \tag{30}$$

3. Solutions of the flow equations

In this section, we seek solutions of the flow equations for various prescribed flow geometries as follows:

Vortex flow

For this flow

$$\eta = r, \quad \zeta = \theta, \quad g_1 = r, \quad g_2 = 1, \quad \alpha = -\theta, \tag{31}$$

Equations (23 – 25, 27, 28), employing equation (31), give

$$q = \Psi'(r) \tag{32}$$

$$-\frac{\Psi'(r\Psi')'}{r} = -L_r + F_1\cos\theta + F_2\sin\theta - \upsilon_\theta R\sin 2\theta - \upsilon R'\sin 2\theta + \frac{\upsilon_\theta R}{r} \tag{33}$$

$$0 = -L_\theta + r(F_1\cos\theta - F_2\sin\theta) - R\upsilon_\theta\sin 2\theta + rR'\upsilon + rR\upsilon_r + \upsilon R(4\sin^2\theta) \tag{34}$$

where

$$R = \Psi'' - \frac{\Psi'}{r} \tag{35}$$

We found that the equations possess a solution when

$$\upsilon = X(r)\,Y(\theta) \tag{36}$$

where X, Y are to be determined.

Equations (33) and (34), on using equation (36) and the integrability condition

$L_{r\theta} = L_{\theta r}$, yield

$$r\cos\theta F_{1r} - r\sin\theta F_{2r} = F_{2\theta}\cos\theta + F_{1\theta}\sin\theta$$

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$$+ Y \{ (r Z')' + 2 Z' \} + \frac{Z Y''}{r} \qquad (37)$$

where

$$Z = R X \qquad (38)$$

A solution of equation (37) is given by

$$F_1 = r \sin\theta$$

$$F_2 = r \cos\theta$$

$$Y = a_2 \theta + a_3 \qquad (39)$$

$$Z = a_1$$

where $a_1 (\neq 0)$, $a_2 (\neq 0)$ and a_3 are arbitrary constants.
Equations (33) and (34), employing equation (39), yield

$$L = \int \frac{\Psi' (r \Psi')'}{r} dr + r^2 - c_3 (a_2 \theta + a_3) \sin 2\theta + a_2 a_1 \ln r + 2c_1 \left(\frac{a_2 \theta^2}{2} + a_3 \theta \right) + c_2 \qquad (40)$$

In equation (40), the streamfunction Ψ is an arbitrary function and $c_1 (\neq 0)$, $c_2 (\neq 0)$, and c_3 are arbitrary constants.
The energy equation (26), in this case, becomes

$$r^2 T_{rr} + r T_r - r \Psi' (\theta) T_\theta + T_{\theta\theta} = - X (a_2 \theta + a_3) (r \Psi''' - \Psi')^2 \qquad (41)$$

The form of the above partial differential equation for T suggests to seek a solution of the form

$$T = H(r) + Q(\theta) \qquad (42)$$

where $H(r)$, $Q(\theta)$ are to be determined. Equation (41), using equation (42), yields

$$r^2 H'' + r H' - r \Psi' Q' + Q'' = - \rho (a_2 \theta + a_3) X (r \Psi''' - \Psi')^2 \qquad (43)$$

Differentiating equation (43) w.r.t. "θ", we get

$$- r \Psi' Q'' + Q''' = - \rho a_2 X (r \Psi''' - \Psi')^2 \qquad (44)$$

Again differentiating equation (44), w.r.t. "θ", we get

$$- r \Psi' Q''' + Q^{(iv)} = 0 \qquad (45)$$

This leads us to the following two cases:

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Case (I) $Q''' = 0$,

Case (II) $Q''' \neq 0$, $r\Psi' = a_4$

We consider these two cases separately

Case (I) When $Q''' = 0$, we get

$$Q = \frac{a_5}{2}\theta^2 + a_6\theta + a_7 \tag{46}$$

where a_5, a_6, a_7 are all non-zero constants
Equation (44), on using equation (46), yields

$$a_3 r\Psi' = \rho a_2 X(r)(r\Psi'' - \Psi')^2 \tag{47}$$

Equation (43) on using equations (46) and (47) yields

$$r^2 H'' + rH' - a_6\frac{\rho a_2 X}{a_3}(r\Psi'' - \Psi')^2 + a_5 = -\rho a_3 X(r\Psi'' - \Psi')^2 \tag{48}$$

whose solution is

$$H = \left(\frac{a_6 a_2}{a_3} - a_5\right) a_1 \rho \int \frac{1}{r}\left[\int (r\Psi'' - \Psi')dr\right]dr - \frac{a_5}{2}[\ln r]^2 + a_8 \ln r + a_9 \tag{49}$$

where a_8 ($\neq 0$) and a_9 are arbitrary constants
Now equations (47) on using the fact that

$Z = RX = a_1$ and $R = (\Psi'' - \frac{\Psi'}{r})$, becomes

$$a_3 r\Psi'' = \rho a_2 a_1 (r\Psi'' - \Psi')r \tag{50}$$

which can be rewritten as

$$\frac{\Psi''}{\Psi'} = \left(1 + \frac{a_3}{\rho a_1 a_2}\right)\frac{1}{r} \tag{51}$$

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whose solution is

$$\Psi = \begin{cases} \dfrac{b_1 r^{2+\frac{a_3}{\rho a_1 a_2}}}{2 + \frac{a_3}{\rho a_1 a_2}} + b_2, & a_3 \neq -2\rho a_1 a_2 \\ b_1 \ln r + b_3, & a_3 = -2\rho a_1 a_2 \end{cases} \tag{52}$$

where $b_1 (\neq 0)$, b_2 and b_3 are arbitrary constants.

Case (II) When $Q''' \neq 0$, $r\Psi' = a_4$, we get

$$\Psi = a_4 \ln r + a_{10} \tag{53}$$

This on using in equation (45), we get

$$Q^{(iv)} - a_4 Q''' = 0 \tag{54}$$

whose solution is

$$Q = -\frac{a_{11}}{2a_4}\theta^2 - \left(\frac{a_{11} + a_4 a_{12}}{a_4^2}\right)\theta + a_{14}e^{a_4\theta} - \left(\frac{a_{11} + a_4 a_{12} + a_4^2 a_{13}}{a_4^3}\right) \tag{55}$$

where $a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$ are all non-zero constants

Equation (54), employing equations (53) and (55), yields

$$a_{11} = -4\rho a_2 a_4^2 \frac{X}{r^2} \tag{56}$$

substituting equations (55) and (56) in (53), we get

$$r^2 H'' + r H' = -\left(a_{12} - \frac{a_{11} a_3}{a_2}\right) \tag{57}$$

whose solution is given by

$$H = -\frac{1}{2}\left(a_{12} - \frac{a_{11} a_3}{a_2}\right)[\ln r]^2 + a_{15} \ln r + a_{16} \tag{58}$$

where $a_{15} (\neq 0)$ and a_{16} are arbitrary constants.

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A particular solution of equation (41) for $a_2 = 0$ is given by

$$T = a_{17}\,\theta + a_{17}\int \frac{\Psi}{r}\,dr + a_{18}\ln r - \rho a_3 \int \frac{1}{r}\left[\int \frac{X\,(r\Psi'' - \Psi')^2}{r}\,dr\right]dr + a_{19} \tag{59}$$

In equation (59) the streamfunction is arbitrary and therefore we can construct a large number of expressions for T and hence a large number of solutions to the flow equations.

If we take $R = \Psi'' - \frac{\Psi'}{r}$ (equation 35) equal to zero, then we obtain

$$\Psi = \frac{m_1 r^2}{2} + m_2 \tag{60}$$

$$F_1 = S_1(\theta) \tag{61}$$

$$F_2 = -\int S_1'(\theta)\tan\theta\,d\theta + m_4 \tag{62}$$

$$L = m_1^2 r^2 + r S_2 \cos\theta + r S_1 \sin\theta + m_3 \tag{63}$$

$$T = \begin{cases} m_5\theta + \dfrac{m_1 m_3}{4} r^2 + m_6 \ln r + m_7, & m_1 m_3 \neq -1 \\ (m_8\theta + m_9)\,J_0(r), & m_1 m_3 = -1 \end{cases} \tag{64}$$

In above equations m_i's are all non-zero constants and $J_0(r)$ is a Bessel function of order zero. The functions $S_1(\theta)$, $S_2(\theta)$ are both arbitrary . The arbitrariness of these functions enables us to construct a large number of solutions to the flow equations.

Radial flow

For radial flows

$$\eta = \theta, \quad \zeta = r, \quad g_1 = 1, \quad g_2 = r, \quad \alpha = 0 \tag{65}$$

and the equations (23 – 28) gives

$$q = \frac{\Psi'(\theta)}{r} \tag{66}$$

$$-\frac{\Psi''}{r^2} = -\frac{L_0}{\Psi'} - \frac{r A_1 \sin 2\theta}{2\Psi'} - \frac{A_0 \cos^2\theta}{\Psi'} + \frac{r B_1 \cos 2\theta}{\Psi'} - \frac{B_0 \sin 2\theta}{\Psi'} + \frac{r}{\Psi'}\,(F_2\cos\theta - F_1\sin\theta) \tag{67}$$

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$$0 = -L_r - A_r \sin^2\theta - \frac{A_\theta \sin\theta\cos\theta}{r} + B_r \sin 2\theta + \frac{B_\theta \cos 2\theta}{r} + F_1 \cos\theta + F_2 \sin\theta \tag{68}$$

$$r^2 T_{rr} + r T_r - r\Psi''(\theta) T_r + T_{\theta\theta} = -\frac{\mu}{r^2}(4W_1^2 + W_2^2) \tag{69}$$

where

$$A = -\frac{4\mu W_1}{\rho r^2} \tag{70}$$

$$B = \frac{\mu W_2}{\rho r^2} \tag{71}$$

$$W_1 = \Psi' \cos 2\theta + \frac{\Psi^*}{2} \sin 2\theta \tag{72}$$

$$W_2 = -2\Psi' \sin 2\theta + \Psi''' \cos 2\theta \tag{73}$$

Substituting

$$\mu = M(\theta) N(r) \tag{74}$$

in equations (67) and (68), we get

$$-\frac{\Psi\Psi^*}{r^2} = -L_0 + \frac{2r}{\rho} Y'(r) W_1^* \sin 2\theta + \frac{4}{\rho} Y W_{1\theta}^* \cos^2\theta + \frac{r}{\rho} Y'(r) W_2^* \cos 2\theta - \frac{1}{\rho} Y W_{2\theta}^* \sin 2\theta + r(F_2 \cos\theta - F_1 \sin\theta) \tag{75}$$

$$0 = -L_r + \frac{4}{\rho} Y'(r) W_1^*(0) \sin^2\theta + \frac{4}{\rho r} Y W_{1\theta}^* \sin\theta\cos\theta + \frac{1}{\rho} Y' W_2^*(0) \sin 2\theta + \frac{1}{\rho r} Y W_{2\theta}^* \cos 2\theta + F_1 \cos\theta + F_2 \sin\theta \tag{76}$$

In equation (75) and (76)

$$W_1^* = M(\theta)\, W_1(\theta)$$

$$W_2^* = M(\theta)\, W_2(\theta) \tag{77}$$

$$Y(r) = \frac{N(r)}{r^2}$$

Equations (75) and (76) suggest

$$L = \frac{U(\theta)}{r^2}, \qquad Y(r) = \frac{1}{r^2} \tag{78}$$

where $U(\theta)$ is an unknown function to be determined.

Substituting equation (78) in equations (75) and (76), we get

$$\begin{aligned} -\psi'\psi'' = -U'(\theta) &- \frac{4}{\rho} W_1^* \sin 2\theta + \frac{4}{\rho} W_{10}^* \cos^2\theta \\ &- \frac{2}{\rho} W_2^* \cos 2\theta - \frac{1}{\rho} W_{20}^* \sin 2\theta \\ &+ r^3 (F_2 \cos\theta - F_1 \sin\theta) \end{aligned} \tag{79}$$

$$\begin{aligned} 0 = \; & 2U(\theta) - \frac{8}{\rho} W_1^* \sin^2\theta + \frac{2}{\rho} W_{10}^* \sin 2\theta \\ &- \frac{2}{\rho} W_2^*(\theta) \sin 2\theta + \frac{1}{\rho} W_{20}^* \cos 2\theta \\ &+ r^3 (F_1 \cos\theta + F_2 \sin\theta) \end{aligned} \tag{80}$$

In the above equations, we note that the only terms that contain F_1 and F_2 are functions of r and θ, and rests are only function of θ. This suggests that we substitute

$$F_1 = \frac{S_1(\theta)}{r^3}, \qquad F_2 = \frac{S_2(\theta)}{r^3} \tag{81}$$

in equations (77), (80), and therefore the equation (80) yields

$$\begin{aligned} U(\theta) = \; & \frac{4}{\rho} W_1^* \sin^2\theta - \frac{1}{\rho} W_{10}^* \sin 2\theta \\ &+ \frac{1}{\rho} W_2^* \sin 2\theta - \frac{1}{2\rho} W_{20}^* \cos 2\theta \\ &- \frac{1}{2} (S_1(\theta) \cos\theta + S_2(\theta) \sin\theta) \end{aligned} \tag{82}$$

Equation (79), on using equations (81), (82) and after laborious algebra, yields

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$$M''\Psi'' + 2M'(\Psi''' + 2\Psi') + M(\Psi^{(iv)} + 4\Psi'') = -2\rho\Psi'\Psi'' - 2\rho(S_2\cos\theta - S_1\sin\theta) - \rho(S_1\cos\theta + S_2\sin\theta)' \quad (83)$$

Treating equation (83) as a differential equation in M, we obtain

$$M(\theta) = Q_1^{-1}(\theta)\left[\int \frac{R_1(\theta)\,Q_1(\theta)}{\Psi''}\,d\theta + d_2\right] \quad (84)$$

where

$$Q_1(\theta) = \exp\left[\int \frac{(\Psi''' + 4\Psi')}{\Psi''}\,d\theta\right] \quad (85)$$

$$R_1 = \int [-2\rho\Psi'\Psi'' - 2\rho(S_2\cos\theta - S_1\sin\theta) - \rho(S_1\cos\theta + S_2\sin\theta)']\,d\theta + d_1\theta \quad (86)$$

and d_1, d_2 are non-zero arbitrary constants.
We mention that in equations (83 – 86) the functions Ψ, S_1 and S_2 are arbitrary. This arbitrariness enables us to construct a large number of expressions for the function $M(\theta)$ and hence for also the functions that involve Ψ, S_1, S_2, $M(\theta)$.

The temperature distribution T for radial flow satisfies the equation

$$r^2T_{rr} + rT_r - r\Psi'(\theta)T_\theta + T_{\theta\theta} = -\frac{\mu}{r^2}(4W_1^2 + W_2^2) \quad (87)$$

A solution of this equation is given by

$$T = \frac{\theta^m}{r^2}\int \theta^{-2m}\left[-\int \mu\theta^m\left(4W_1^2 + W_2^2\right)d\theta + d_4\right]d\theta + d_5 \quad (88)$$

Provided

$$\Psi = \frac{m(m-1)}{2\theta} - \frac{\theta}{2} + d_3, \qquad m \neq 1 \quad (89)$$

In equation (88), $\mu = M(\theta)$ the function is given by equation (83) which involves arbitrary functions $S_1(\theta)$, $S_2(\theta)$ and this arbitrariness of S_1, S_1 enables us to generate a large number of temperature distributions.

Parallel flow

For this type of flow

$$\zeta = x, \qquad \eta = y, \qquad g_1 = 1 = g_2, \qquad a = 0 \quad (90)$$

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Equations (23 – 28), in using equation (90), yield

$$L_x = F_1 + \frac{1}{\rho}(\mu\Psi''(y))_y \tag{91}$$

$$L_y = \Psi'(y)\Psi''(y) + F_2 + \frac{\Psi''(y)}{\rho}\mu_x \tag{92}$$

$$T_{xx} + T_{yy} = -\mu\Psi''^2(y) + \Psi'(y)T_x \tag{93}$$

Employing the compatibility conditions $L_{xy} = L_{yx}$, we get

$$F_{1y} + \frac{1}{\rho}(\mu\Psi'')_{yy} = F_{2x} + \frac{\Psi''}{\rho}\mu_{xx} \tag{94}$$

Substituting

$$\mu = X(x)Y(y) \tag{95}$$

in equation (94), we obtain

$$F_{1y} + \frac{1}{\rho}XZ''(y) = F_{2x} + \frac{Z(y)}{\rho}X_{xx} \tag{96}$$

where

$$Z(y) = \Psi''(y)Y(y) \tag{97}$$

The equation (96) suggests to take

$$F_1(x, y) = F_{11}(x)F_{12}(y)$$
$$F_2(x, y) = F_{21}(x)F_{22}(y) \tag{98}$$

Equation (96), utilizing equation (98), can be written as

$$\frac{F_{11}(x)}{X(x)}\frac{F_{12}'(y)}{Z(y)} + \frac{1}{\rho}\frac{Z''(y)}{Z(y)} = \frac{F_{11}'(x)}{X(x)}\frac{F_{12}(y)}{Z(y)} + \frac{X_{xx}(x)}{\rho X(x)} \tag{99}$$

Differentiating equation (99) w.r.t. "y", we get

$$\left(\frac{F_{11}}{X}\right)\left(\frac{F_{12}'}{Z}\right)' + \frac{1}{\rho}\left(\frac{Z''}{Z}\right)' = \left(\frac{F_{21}'}{X}\right)\left(\frac{F_{22}}{Z}\right)' \tag{100}$$

Differentiating equation (100) w.r.t. "x", we get

$$\left(\frac{F_{11}}{X}\right)'\left(\frac{F_{12}'}{Z}\right)' = \left(\frac{F_{21}'}{X}\right)'\left(\frac{F_{22}}{Z}\right)' \tag{101}$$

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which can be written as

$$\frac{\left(\frac{F_{11}}{X}\right)'}{\left(\frac{F_{21}'}{X}\right)'} = \frac{\left(\frac{F_{22}}{Z}\right)'}{\left(\frac{F_{12}'}{Z}\right)'} \tag{102}$$

in equation (102) the terms in denominators are non-zero.
Since equation holds for all x and y, therefore

$$\frac{\left(\frac{F_{11}}{X}\right)'}{\left(\frac{F_{21}'}{X}\right)'} = \frac{\left(\frac{F_{22}}{Z}\right)'}{\left(\frac{F_{12}'}{Z}\right)'} = \text{constant} = n_1 \text{ (say)}. \tag{103}$$

This gives

$$F_{11} = n_1 F_{21}' + n_2 X \tag{104}$$

$$F_{22} = n_1 F_{12}' + n_3 Z \tag{105}$$

where n_1, n_2, n_3 are all non-zero constants.
Equation (100), employing equations (104) and (105), yields

$$Z'' + \rho n_2 F_{12}' = n_4 Z(y) \tag{106}$$

Employing equations (104 – 106) in equation (99), we get

$$X_{xx} - n_5 \rho X = -\rho n_1 F_{11}' \tag{107}$$

$$n_4 = \rho n_5 \tag{108}$$

where n_5 is an arbitrary constant.
We now determine the solution of equations (107) as follows.
When $n_5 = 0$, the solution of equation (107) is given by

$$X = n_6 x + n_7 - \rho n_1 \int F_{21}\, dx \tag{109}$$

When $n_5 > 0$, then

$$X = n_8 e^{\sqrt{\rho n_5}\, x} + n_9 e^{-\sqrt{\rho n_5}\, x} + \frac{n_1\sqrt{\rho}}{2\sqrt{n_5}} e^{-\sqrt{\rho n_5}\, x} \int e^{\sqrt{\rho n_5}\, x} F_{21}'\, dx - \frac{n_1\sqrt{\rho}}{2\sqrt{n_5}} e^{\sqrt{\rho n_5}\, x} \int e^{-\sqrt{\rho n_5}\, x} F_{21}'\, dx \tag{110}$$

When $n_4 < 0$, then

$$X = n_{11}\cos(\sqrt{\rho n_4}\, x) + n_{12}\sin(\sqrt{\rho n_4}\, x) + \frac{n_3\sqrt{\rho}}{\sqrt{n_4}}\cos(\sqrt{\rho n_4}\, x)\int \sin(\sqrt{\rho n_4}\, x)\, F_{21}\, dx - \frac{n_3\sqrt{\rho}}{\sqrt{n_4}}\sin(\sqrt{\rho n_4}\, x)\int \cos(\sqrt{\rho n_4}\, x)\, F_{21}\, dx \quad (111)$$

The solution of equation (106) for n_4 ($= 0, > 0, < 0$) as follows
When $n_4 = 0$, the solution is

$$Z = -\rho n_2 \int F_{12}\, dy + n_{13} y + n_{14} \quad (112)$$

When $n_4 > 0$, then

$$Z = n_{15} e^{\sqrt{n_4}\, y} + n_{16} e^{-\sqrt{n_4}\, y} - \frac{n_2\sqrt{\rho}}{2\sqrt{n_4}} e^{-\sqrt{n_4}\, y}\int e^{\sqrt{n_4}\, y} F_{12}\, dy - \frac{n_2\sqrt{\rho}}{2\sqrt{n_4}} e^{\sqrt{n_4}\, y}\int e^{-\sqrt{n_4}\, y} F_{12}\, dy \quad (113)$$

When $n_4 < 0$, $\rho n_4 = -n_{17}^2$ and $n_{17} \neq 0$, then

$$Z = n_{17}\cos(n_{17} y) + n_{18}\sin(n_{17} y) - \frac{n_2\rho}{n_{17}}\cos(n_{17} y)\int \sin(n_{17} y)\, F_{12}\, dy - \frac{n_2\rho}{n_{17}}\sin(n_{17} y)\int \cos(n_{17} y)\, F_{12}\, dy \quad (114)$$

In equations (112 - 114), n_{13}, n_{14}, n_{15}, n_{16}, n_{17}, n_{18} are all non-zero constants, and F_{12} is an arbitrary

We note here that the components F_{11}, F_{12}, F_{21}, F_{22} of the external force are all arbitrary. When in equation (101)

$$\left(\frac{F_{12}}{Z}\right)' = 0, \qquad \left(\frac{F_{21}}{X}\right)' = 0 \quad (115)$$

we find that

$$F_{21} = n_{19} \int X\,dx + n_{21}$$

$$F_{12} = n_{20} \int Z\,dy + n_{22} \qquad (116)$$

Now equation (101) is identically satisfied on using equation (115) and the equation (99) gives

$$F_{11} = \frac{X}{n_{20}} \left(\frac{X'}{\rho X} - n_{23} \right)$$

$$F_{22} = \frac{Z}{n_{19}} \left(\frac{Z'}{\rho Z} - n_{23} \right) \qquad (117)$$

We mention that in this case X, Z, F_{11}, F_{21} are all arbitrary.
The temperature distribution T satisfies the equation (93) and its solution can easily be obtained by substituting expressions for μ and $\Psi(y)$. For the above case if we take X = 1, then equation (93) becomes

$$T_{xx} + T_{yy} = -Y(y)\Psi''^{2}(y) + \Psi''(y)T_x \qquad (118)$$

A solution of equation (118) is given by

$$T = n_{24}x + \int \left[\int (-Y\Psi'' + n_{24}\Psi')dy \right] dy + n_{25}y + n_{26} \qquad (119)$$

In equation (119), the functions Y, Ψ are both arbitrary, and their arbitrariness enables us to construct a large number of solutions to the flow equations.
For parallel flows the energy function L is given by

$$L = \frac{\Psi'^2}{2} + F_{21}(x) \int F_{22}(y)\,dy + \frac{X_0}{\rho} \int Z\,dy + n_{27} \qquad (120)$$

where n_{27} is an arbitrary constant.

4. Conclusions

In the present work the equations governing the steady plane motion of an incompressible fluid of variable viscosity and constant thermal conductivity in the presence of unknown external force for an arbitrary state equation with no heat addition are considered. The equations are transformed into a new system of equations in an orthogonal coordinate system (Φ, Ψ) in which Ψ = constant represent streamline and Φ = constant are left as a family of arbitrary curves. Exact solutions to the flow equations are determined for radial, vortex and parallel flows. The expressions for the external force are also determined for each flow. The solutions involve arbitrary functions enabling to construct a large number

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