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ON IDENTICAL BLOCKS IN INCOMPLETE BLOCK DESIGNS

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Abstract

Number of identical blocks in any linery equireplicate, equiblock–size incomplete block designs are obtained. The results are applied to find the number of identical blocks in some known designs.

Keywords: Identical blocks, known designs.

Introduction

The number of common treatments between any two blocks in a Balanced incomplete block design is given by [1]. The numbers of common treatments between two blocks in certain PBIBD's is obtained by [3]. Repetitions of some blocks in a design is an interesting combinatorial property. The number of identical blocks in a BIBD was obtained by [2]. The problem of identical blocks in a BIBD with parameters (v, b, r, k) = (2x +2, 4x + 2, 2x +1, x + 1, x) was considered by [7].

In this paper we derive bounds for number of identical blocks in any BIBD and use this general result to obtain the number of identical blocks in some known design.

2. Number of identical blocks in an incomplete block design:

We consider a binary equireplicate equiblock-sized IBD (v, b, r, k,) let N be the incidence matrix of the given design.

Let $rk > \theta_0 > \theta_1 > \ldots\ldots > \theta p$ denote the distinct non zero

Characteristics roots of NN' with respective multiplicities ﻙ,

ﻙ $_0$, ﻙ $_1$ ……..ﻙ þ

Then, the non zero characteristic roots of NN' are $rk > \theta o > \theta_1 > \ldots\ldots > \theta p$. It is easy to verify that the non-zero characteristics roots of the matrix are

$$N'N - \frac{rk - \theta}{b} \quad Ebb \text{ ——— } \quad (2.1)$$

are $\theta o > \theta_1 > - - - > \theta p$.

$$\text{Then } Y' N \; N'Y - \frac{rk - \theta o}{b} Y' EbbY \leq \theta o \, Y'Y \text{--------} \quad (2.2).$$

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We suppose that there are d identical blocks, which are denoted by B1. B2. . . .B1 . . . . Bd. The remaining (b-d) blocks are denoted by Bd+1. . . . Bd+2. . . . . Bb. Let lij denote the number of treatments common between the blocks Bi and Bj clearly.

$$\ell ij=k \quad i \neq j = 1,2, \ldots d \qquad (2.3)$$

Selecting the vector y as

$$y' = [\underbrace{d^{-½}, d^{-½} \ldots d^{-½}}_{d}, \underbrace{0, \ldots . 0}_{b-d}]$$

In (2.2) we obtain $$d \leq \frac{b\theta o}{k(b-r)+\theta o} \qquad (2.4)$$

Since $d \geq 1$, we obtain $1 \leq d \leq b\theta o \setminus K(b-r)+\theta o$ (2.5)

Thus we have the following theorem:

Theorem 2.1 let N be the incidence matrix of an IBD (v, b, r, k) and $rk > \theta o > \theta_1 > \ldots . . > \theta p$ be the distinct non zero characteristic roots of NN'. If d denotes the no. of identical blocks in the design. Then it satisfies.

$$1 \leq d \leq b\theta_0 \setminus [k(b-r)+\theta o].$$

We now consider one simplification. We suppose the design is a connected one, so that the characteristic root rk has multiplicity 1. We put all the characteristic roots other then rk equal to zero, except one ch: root, θ, say with multiplicity ؘ .Then since the sum of ch: roots of a matrix is equal to its trace, we get

$$rk + ؘ\theta = tr(NN') = vr,$$

which gives $\theta = K(b-r) \mid ؘ$ .Using Th: 2.1 we obtain the following theorem.

Theorem 2.2 let θ be the only non zero ch: root of NŃ other than rk, of multiplicity ؘ , f a connected IBD (v, b , r, k,) and d the no. of identical blocks in the design . Then

i) $1 \leq d \leq b \mid (ؘ + 1)$

ii) if $d = b \mid (ؘ + 1)$ then each of the remaining blocks contains $\frac{k[r(ؘ+1) - b]}{bؘ}$ treatments in common with each identical block, and

iii) if $b < 2(\delta+1)$ then no two blocks are identical we should now consider the application of Th: 2.2 to some known designs.

3. Applications

In a BIBD the ch: roots of NN' are rk and r-x with multiplicities 1 and V–1 respectively. Hence in this case $\theta = r - \lambda$ and δ V-1. Then applying Th: 2.2 one can easily derive the results due to Mann (1969). Thus Th: 2.2 provides alternative proof of known results, for a BIBD, We now consider the applications of Th: 2.2 to group divisible, triangular, Li and rectangular designs.

1. Group divisible designs. In a group divisible (GD) design v = mn treatments are divided into m groups of n treatments each such that any two treatments belonging to the same group are first associates and two treatments belonging to different groups are second associates. The Ch: roots of NN' for a GD designs are rk, $r-\lambda_1$, $rk - v\lambda_2$ with respective multiplicities 1,v – m, m – 1 [6]. The GD designs are classed as (i) Singular GD (ii) Semi-regular GD and (iii) regular GD, according as (i) $r - \lambda_1 = o$ (ii) $r - \lambda_1 > o$ and $rk - v\lambda_2 = o$ and (iii) $r-\lambda_1 = o$ and $rk-v\lambda_2$. Using Th: 2.2 for singular and semi-regular GD designs Th: 3 let d be the no. of identical blocks in a singular GD design.

Then

i) $1 \leq d \leq b \setminus m$ ii) if $d = b \setminus m$, then each of the remaining blocks contain $k [rm - b] \setminus b$ (m-1) treatments in common with each identical block and (iii) if $b < 2m$ then, no two blocks are identical.

Th : 3.2 = let d be the no: of identical blocks in a semi-regular GD design.

Then

i) $1 < d \leq b / (v-m+1)$

ii) if $d = b/ (v-m+1)$ then each of the remaining blocks contain $k [\frac{r(v-m+1)- b}{b}]$

(v-m)treatments in common with each identical block and (iii) if $b<2$ (v-m+1), then no two blocks are identical. Consider a semi-regular GD design in which b = v-m + r and v =2k. Cl early b=2r and hence from b = v-m + r we get b= 2 (v-m). Thus we get $b<2(v- m + r)$.

ii) Triangular designs. For the description of a triangular design [5] the no. of treatments is $v = n(n-1)/2$. The ch: roots of NN' of this design are rk, $r + (n-4)\lambda_1 + (n-3)\lambda_2$ and $r-2\lambda_1 + \lambda_2$ with receptive multiplicities 1,n-1 and v-n. A triangular design with $r + (n-4)\lambda_1 - (n-3)\lambda_2 = 0$ has some interesting block structure properties [6]. We therefore consider this particular triangular design we get.

Th : 3.3 : let d denote the no. of identical blocks in a triangular design with $r + (n-4)\lambda_1 - (n-3)\lambda_2 = o$ then (i) $1 < d < b/(v-n+1)$ (ii) if $d = b/(v-n+1)$ then each of the remaining blocks contains k $[r(v-n+1)/b(v-n)$ treatments in common with every identical block and (iii) $b < 2(v-n+1)$ then no two blocks are identical.

Consider a triangular design with $r + (n-4)\lambda_1 - (n-3)\lambda_2 = 0$ $b = v-n+r$ and $u = 3k$. since $v = 2k$,we have $b=2r$. Hence from $b = v-n+r$, we get $b=2(v-n)$. thus we have $b<2(v-n+1)$.

(iii) Li design. For the Li designs [5] the no. of treatments is $v = s^2$. The ch: roots of NN' of this design are rk, $r + (s-i)\lambda_1 - (s-i+1)\lambda_2$ and $r-i\lambda_1 + (i-I)\lambda_2$ with respective multiplicities 1, i (s-1) and (s-i+1) (s-1) . The l i design with $r + (s-i)\lambda_1 - (s-i+1)\lambda_2 = 0$ has some intresting block structure properties [6]. Hence we consider this particular Li design. Applying Th: 2.2 for this particular design, we get

Th: 3.4 let d denote the no: of identical blocks in a li design with $r + (s-i)\lambda_1 - (s-i+1)\lambda_2 = 0$ then

i. $1 \le d \le b/[(s-i+I)(s-1)+1]$,

ii. if $d = b/[(s-i+1)(s-1)+1]$, then each of the remaining blocks contains k $[r\{(s-i+1)(s-1)+1\} - b]/b(s-i+1/(s-1)$ treatments in common with each identical block and

iii. if $b < 2(s-i+1)(s-1) + 2$, then no two blocks are identical, Taking i = 2, i.e for a L_2 design.

iv. Rectangular Design. For the description of rectangular design we refer to [7]. In a rectangular design the no. of treatments is v = mn and the treatments are arranged in a rectangular of m rows and n columns. Th ch: roots of NN' of this design are rk. $r-\lambda_1 + (m-1)(\lambda_2-\lambda_3)$, $r-\lambda_2 + (n-1)(\lambda_1 - \lambda_3)$ and $r-\lambda_1-\lambda_2+\lambda_3$ with respective multiplicities 1,n-1, m-1 and (n-1) (m-1). The rectangular design with $r-\lambda_1+(m-1)(\lambda_2-\lambda_3)=0$ and $r-\lambda_2 + (n-1)(\lambda_1-\lambda_3) = 0$ will be considered here.

Applying Th : 2.2 to this particular design, we get

Th : 3.5. let d denote the no. of identical blocks in a rectangular design with $r-\lambda_1 + (m-1)(\lambda_2-\lambda_3)=0$ and $r-\lambda_2 + (n-1)(\lambda_1-\lambda_3) =0$ then

i. $1 \leq d \leq b/[(n-1)(m-1)+1]$

ii. if $d = b/[(n-1)(m-1)+1]$, then each of the remaining blocks contains $k[r\{(n-1)(m-1)+1\}-b]/(n-1)(m-1)$ treatments in common with every identical block .and

iii. if $b < 2(n-1)(m-1)+2$, then no two blocks are identical. Consider a rectangular design in which $r - \lambda_1 + (m-1)(\lambda_2 - \lambda_3) = 0$, $r - \lambda_2 + (n-1)(\lambda_1 - \lambda_3) = 0$, $b = (n-1)(m-1) + r$ and $v= 2k$, since $v=2k(n-1)(m-1)$. We have $b = 2r$. hence from $b = (n-1)(m-1) + r$ we get $b=2(n-1)(m-1)$. Thus we have $b<2(n-1)(m-1) + 2$.

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