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PARALLEL MATRIX MULTI-SPLITTING METHODS FOR SYSTEM OF WEAKLY NON-LINEAR EQUATIONS

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Abstract

In this paper, a class of parallel matrix multisplitting algorithms is set up to approximate the unique solution of weakly non-linear algebraic system of equations $A u = G(u)$, where $A \in L(R^n)$ is an H-matrix & $G(u) : R^n \to R^n$, $u \in R^n$, $G(u)$ being a nonlinear mapping.

Key Words: Matrix multisplitting, Nonlinear mapping, H-matrix, Relaxation parameters M-Splitting & P-Splitting.

1. Introduction

Consider the weakly nonlinear algebraic system of equations:

$$A u = G(u),\ A \in L(R^n),\ \text{nonsingular},\ G(u) : R^n \to R^n,\ u \in R^n \qquad (1.1)$$

Iterative methods for approximation the unique solution of (1.1) are based on multisplitting of the coefficient matrix A, Multisplitting method first proposed by O'Leary and White [1] where several basic convergence results may be found, Neumann and Plemmons [2] developed some more refined convergence results for one of the cases considered in [1]. It has already been seen in [1] that the introduction of a relaxation parameter may considerably improve the multisplitting methods, but recently Formmer A. and G.Mayer in [3] published convergence results for these modifications of multisplitting methods as well as Wang Deren in [4], presented the convergence of the parallel multisplitting AOR method. In the present paper we set up the convergence of a class of parallel weakly nonlinear relaxation algorithms under certain restrictions provided $A \in L(R^n)$, is an H-matrix is also discussed.

2 Preliminaries

Some definitions, notations and lemmas are standard we frequently borrow from the Language of O'Leary and White in [1], Formmer A. and G.Mayer in [3] and Wang Deren in [4], as used in their papers 1985, 1989 and 1991 respectively.

Definition 2. 1:

Let $A \in L(R^n)$, nonsingular, matrix & $(B_k, C_k, E_k) \in R^{n\times n}$, $k = 1(1)\alpha$, verifying

(a) $A = B_k - C_k$,

(b) $\det(B_k) \neq 0$, B_k is nonsingular,

(c) E_k is a nonnegative diagonal matrix, $(E_k \geq 0,\ \sum_{k=1}^{\alpha} E_k = I \in L(R^n)$

Identity matrix for $k = 1(1)\alpha$

Then the collection of ordered triples (B_k, C_k, E_k), $k = 1(1)\ \alpha$ is called a multisplitting of A.

Definition 2. 2:

Let $u \in R^n$, we can say $u \geq 0\ (> 0)$ if its components $u_i \geq 0\ (> 0)$, $i = 1(1)n$.

Let $A, B \in L(R^n)$, we can write $A \leq B$ if $a_{i\,j} \leq b_{i\,j}$ holds for all entries of $A = (a_{i\,j})$ and $B = (b_{i\,j})$, calling $A \in L(R^n)$ non-negative if $A \geq 0$. By $|A| = (|a_{i\,j}|)$, we can define the absolute value of $A \in L(R^n)$, it is a non-negative matrix satisfying $|A\,B| \leq |A|\,|B|$.

We can denote by $< A > = (<a_{i\,j}>)$ the comparison matrix (Ostrowski matrix) of $A \in L(R^n)$, where,

$$<a_{i\,j}> = \begin{cases} |a_{i\,j}| & \text{if } i = j \\ |b_{i\,j}| & \text{if } i \neq j \end{cases}$$

we can call $A = (a_{i\,j}) \in L(R^n)$ is a M-matrix if it is nonsingular with $a_{i\,j} \leq 0$ for $i \neq j$ and with $A^{-1} \geq 0$, we can call it an H-matrix if $<A>$ is an M-matrix , we can call $A = (a_{i\,j}) \in L(R^n)$ is an L-matrix if $a_{i\,j} > 0$ $(I = 1(1)n)$ and $a_{i\,j} \leq 0, I \neq j$, $(i, j = 1(1)n)$. The splitting of $A \in L(R^n)$ into:

$$A = B - C,\ \det(B) \neq 0 \qquad (2.3)$$

If $C \geq 0$ and $B^{-1} \geq 0$, then (2.3) is called a regular splitting while if $B^{-1}C \geq 0$ and if $B^{-1} \geq 0$, then (2.3) is called weak regular splitting. If B is an M-matrix and $C \geq 0$ then (2.3) is called an M-splitting (P-splitting). For more detail background and non-negative matrices , monotone matrices and related topics in [5], [6] and [7].

LEMMA 2.4: Let $A \in L(R^n)$ be an H-matrix $\operatorname{diag}(A) = D = \operatorname{diag}(B_k)$, $k = 1(1)\alpha$ and $<A> = |D| - |B|$ if $\det(|D|) \neq 0$. Let $J = |D|^{-1}|B|$ then $<A> = |D|\,(I - J)$ and $<A>^{-1} = (I - J)^{-1}|D|^{-1}$.

PROOF: [3]

LEMMA 2.5: Let $A \in L(R^n)$ be a non-negative irreducible matrix. Then the spectral radius of A $(\rho(A))$ is an eigenvalue and eigenvector u corresponding $\rho(A)$ satisfies $u > 0$.

PROOF: [7]

LEMMA 2.6: Let $A, B \in L(R^n)$ be M-matrices $\operatorname{diag}(A) = D = \operatorname{diag}(B_k)$, $k = 1(1)\ \alpha$ and $C \in R^{n \times n}$ then

(a) $\det(D) \neq 0$, D, D^{-1} are nonsingular matrices with positive diagonal entries

(b) $A \leq B \Rightarrow B^{-1} \leq A^{-1}$,

(c) $A \leq C \leq D \Rightarrow C$ is an M-matrix.

PROOF: ([8], 2.4.8 and 2.4.9).

LEMMA 2.7: Let A, B ∈ L (R^n) such that |A| ≤ B, then ρ (A) ≤ ρ(B).
PROOF: [3].

LEMMA 2.8: Let A ∈ L(R^n) be an H-matrix diag (A) = D = diag (B_k), k = 1(1) α and <A> = |D| - |B|, then
(a) A is nonsingular (det (A) ≠ 0),
(b) $|A^{-1}| \leq <A>^{-1}$,
(c) |D| is nonsingular (det (|D|) ≠ 0).
PROOF: [3].

3 Establishment of parallel algorithms:

Recall definition (2.1), the multisplitting method to solve (1.1) is defined by the iteration

$$\begin{cases} u^{m+1} = \sum_{k=1}^{\alpha} E_k u^{m,k}, & m = 0, 1, \ldots \\ u^{m,k} = B_k^{-1} C_k u^m + B^{-1} G(u^m), & k = 1(1)\ \alpha \end{cases} \tag{3.1}$$

The iteration (3.1) can be written in the form

$$\begin{cases} u^{m+1} = H u^m + Q G(u^m), & m = 0, 1, \ldots \\ H = \sum_{k=1}^{\alpha} E_k B_k^{-1} C_k, \quad Q = \sum_{k=1}^{\alpha} E_k B_k^{-1} \end{cases} \tag{3.2}$$

The calculation $u^{m,k}$ for various k are independent and can therefore be performed in parallel. Moreover, considerable savings in computational work may be possible, since a component of $u^{m,k}$ needs not be computed if the corresponding diagonal entry of E_k is zero. The role of the matrices E_k may be regarded as determining the distribution of the computational work to the individual processors. For details in [1]. Note that the above idea of breaking up a problem into several parallel parts has meanwhile given rise to multisplitting methods for system of nonlinear equations [5, 9] and for system of linear equations [3].

In [1] and in [3] also considered a relaxed multisplitting method in which a positive relaxation parameter is used in the same manner as in the relaxed Jacobi method. This method is thus given by the iteration.

$$\begin{cases} u^{m+1} = \omega F_k (u^m) + (1 - \omega) u^m \\ \qquad\qquad\qquad\qquad m = 0, 1, \ldots \\ F_k (u^m) = H u^m + Q G(u^m) \end{cases} \tag{3.3}$$

Another possibility to introduce a relaxation parameter arises if all the matrices B_k, k = 1(1)α are of a particular lower triangular form. It is convenient to use a different notation in this case, suppose that we have,

$$\begin{cases} A = D - L_k - U_k, \\ D = \text{diag}(A) = \text{diag}(B_k), \qquad \text{for } k = 1(1)^{\alpha} \\ L_k \text{ is strictly lower triangular,} \\ U_k \text{ is in general not upper triangular.} \end{cases}$$

We consider the collection of triples $(D - L_k, U_k, E_k)$, $k = 1(1)\alpha$ in the multisplitting of A. Then the calculation of the intermediate results u^m will be done by solving the lower triangular system:

$$\begin{cases} (D - L_k)\, u^{m,\,k} = [(1 - \omega)\, D + (\omega - r)\, L_k + \omega\, U_k]\, u^m + \omega\, G(u^m), \\ \qquad\qquad r \geq 0,\ \omega > 0,\ k = 1(1)\ \alpha \\ \text{The iteration can be written in this form} \\ u^{m+1} = \sum_{k=1}^{\alpha} E_k\, u^{m.k}, \quad m = 0, 1, \ldots \end{cases}$$

The convergence of the above methods will prove in the later section.

Existence Uniqueness and Equivalence of the problem.

THEOREM 3.5: Let $A \in L(R^n)$ be an H-matrix and $(B_k\ C_k\ E_k)$, $k = 1(1)\ \alpha$ be multisplitting of A with $\text{diag}(A) = D = \text{diag}(B_k)$, for $k = 1(1)\ \infty$ and let $\langle A \rangle = \langle B_k \rangle\, |C_k|$, for $k = 1(1)\ \alpha$ Let

(a) $H \geq 0$,

(b) $L\,A = I - H$, Then L^{-1} exist if $\rho(|H|) < 1$.

PROOF: We will only show that $\rho(|H|) < 1$, where H, is the matrix given by (3.2)

$$|H| = \left| \sum_{k=1}^{\alpha} E_k\, B_k^{-1}\, C_k \right|$$

$$\leq \sum_{k=1}^{\alpha} E_k \left|B_k^{-1}\right| |C_k|$$

$$= \sum_{k=1}^{\alpha} E_k \left|B_k^{-1}\right| (\langle B_k \rangle - \langle A \rangle)$$

By lemmas (2.4) and ((2.6)c), $\langle A \rangle = |D|\,(I - J)$ and $|D^{-1}| \leq \langle B_k \rangle$ inequality holds

$$\leq I - \sum_{k=1}^{\alpha} E_k \left|D^{-1}\right| |D|\, (I - J)$$

Let e denote the $e = (1, 1, \ldots, 1)^T \in R^k$. Since J is non-negative $(J \geq 0)$.

the matrix $J + \varepsilon\, e\, e^T$ has only positive entries and is irreducible of any $\varepsilon > 0$. By the theorem of Perron and Frobenius [8], for any $\delta > 0$, there is a vector $x_\varepsilon > 0$ such that $(J + \varepsilon\, e\, e^T)\, x_\varepsilon = \rho_\varepsilon\, x_\varepsilon$ where $\rho_\varepsilon = \rho(J + \varepsilon\, e\, e^T)$, we have $\rho_\varepsilon < 1$.

$$|H| \leq I - \sum_{i=1}^{n} E_i\, |D^{-1}|\, |D|\, (I - (J + \varepsilon\, e\, e^T)$$

$$= I - I + (J + \varepsilon\, e\, e^T)$$

Multiplying x_ε to both sides

$$|H|\, x_\varepsilon \leq (J + \varepsilon\, e\, e^T)\, x_\varepsilon$$

$$|H|\, x_\varepsilon = \rho_\varepsilon\, x_\varepsilon$$

since $\rho_\varepsilon < 1$

$$\frac{(|H|\, x_\varepsilon)_i}{(x_\varepsilon)_i} < 1, \text{ for } i = 1(1)n$$

Hence $\dfrac{(|H|\, x\varepsilon)i}{(x_\varepsilon)_i} < 1$, for i = 1(1)n & ex. 2 in [10, P.47] guarantees $\rho(|H|) < 1$

Let x^* be the unique solution of (1.1) which satisfies $A\, x^* = G(x^*)$ and x^* is also the unique solution of

$$x = H\, x + Q\, G(x) \tag{3.5}$$

which satisfies the

$$x^* = H\, x^* + Q\, G(x^*) \tag{3.6}$$

since $x = H\, x + Q\, G(x)$, $(I - H)\, x = L\, G(x)$, since Q^{-1} exists, consequently $A\, x = G(x)$.

Hence $A\, x = G(x)$ is equivalent to (3.7) therefore x^* is also unique solution of (3.7).

Subtract (3.8) from (3.2) and theorem (3.18(b)), we obtain

$$\leq ((|H|\ |Q|\ P)\ |x^m - x^*|) \tag{3.7}$$

$x^* \in R^n$ is the unique solution of

$$x = \omega\, (H\, x + Q\, G(x)) + (1 - \omega)\, x \tag{3.8}$$

which satisfy

$$x^* = \omega\, (H\, x^* + Q\, G(x^*)) + (1 - \omega)\, x^* \tag{3.9}$$

subtract (3.11) from (3.3) and theorem (3.18(b)), we get

$$|x^{m+1} - x^*| \leq (\omega\, (|H|\ |Q|\ P) + |1 - \omega|\ I)\, |x^m - x^*| \tag{3.10}$$

$$\varsigma = \omega\, (|H|\ |Q|\ P) + |1 - \omega|\ I \tag{3.11}$$

$x^* \in R^n$ is the unique solution of

$$x = (D - r\, L_k)^{-1}\, \{\, [\, (1 - \omega)\, D + (\omega - r)\, L_k + \omega\, U_k]\, x + \omega\, G(x)\} \tag{3.12}$$

which satisfy

$$x^* = (D - r L_k)^{-1} \{ [(1 - \omega) D + (\omega - r) L_k + \omega U_k] x^* + \omega G(x^*) \} \quad (3.13)$$

subtract (3.15) from (3.4) and theorem (3.18(b)), we get

$$|x^{m+1} - x^*| \leq \sum_{k=1}^{\alpha} E_k \, | (D - r L)_k | \, [\, |1 - \omega| \, |D| + (\omega - r) |L_k| \, \omega |U_k| + \omega P] \, |x^m - x^*| \quad (3.14)$$

$$\varsigma = \sum_{k=1}^{\alpha} E_k \, | (D - r L) | \, [\, |1 - \omega| \, |D| \, | \, (\omega - r) |L| \, \omega |U| + \omega P] \quad (3.15)$$

The two multisplitting methods (3.3) and (3.4) are the methods will deal with in the present paper convergence of (3.3) and (3.4) equivalent to $\rho(\varsigma) < 1$ where the matrix ς is defined by (3.13) and (3.15) respectively.

THEOREM 3.18: Let $A \in L(R^n)$ be an H-matrix and (B, C, E), $k = 1(1)\ \alpha$ multisplitting of A with $\text{diag}(A) = D = \text{diag}(B_k)$, $k = 1(1)\ \alpha$. Let

(a) $\langle A \rangle = \langle B_k \rangle - \langle C_k \rangle = |D| - |B|$, $k = 1(1)\ \alpha$

(b) $|G(x) - G(y)| \leq P\,|x - y|$, $0 \leq P \in L(R^n)$, $x, y \in R^n$

(c) $\rho(A^{-1} P) < 1$

Then the method (3.2) converges to the unique solution to x^* for starting point $X^0 \in R^n$

PROOF: We only need to prove $\rho(|H| + |Q| P) < 1$

$$(|H| + |Q| P) = (| \sum_{k=1}^{\alpha} E_k \, B^{-1} \, C_k | + | \Sigma \, E_k \, B^{-1} \, | P)$$

$$\leq \sum_{k=1}^{\alpha} E_k \, |B^{-1}| \, |C_k| + \Sigma \, E_k \, |B^{-1}| \, P$$

since, $\langle C_k \rangle = \langle B_k \rangle - \langle A \rangle$

$$\leq \sum_{k=1}^{\alpha} E_k \, |B^{-1}| \, (\langle B_k \rangle - \langle A \rangle) + \sum_{k=1}^{\alpha} E_k \, |B^{-1}| \, P$$

By lemma (2.8(b)) $|B^{-1}| \leq \langle B_k \rangle^{-1}$ inequality holds

$$\leq \sum_{k=1}^{\alpha} E_k \langle B_k \rangle^{-1} (\langle B_k \rangle - \langle A \rangle) + \sum_{k=1}^{\alpha} \langle B_k \rangle^{-1} P$$

$$\leq I - \sum_{k=1}^{\alpha} E_k \langle B_k \rangle^{-1} \langle A \rangle \, (I - \langle A \rangle^{-1} P)$$

By lemma (2.4) and (2.6(c))

$\langle A \rangle = |D|\,(I - J)$, $\langle A \rangle^{-1} = (I - J)^{-1}\ |D|^{-1}$ and $|D|^{-1} \le \langle B_k \rangle^{-1}$, inequality holds respectively.

$$\le I - \sum_{k=1}^{\alpha} E_k\, |D|^{-1}\, |D|\, (I - J)\, (I - (I - J)^{-1}\, |D|^{-1}\, P)$$

$$= I - (I - J) + (I - J)\, (I - J)^{-1}\, |D|^{-1}\, P$$

$$(|H| + |Q|\, P) = I - J + J + |D|^{-1}\, P$$

By lemma (2.4) $J = |D|^{-1}\, |B|$

$$\le |D|^{-1}\, |B| + |D|^{-1}\, P$$

$$= |D|^{-1}\, (|B| + P)$$

$$\rho(|H| + |Q|\, P) \le \rho(|D|^{-1}\, (|B| + P)) < 1$$

Therefore,
$\rho(|H| + |Q|\ P) < 1$

4 Convergence results:

THEOREM 4.1: Let $A \in L(R^n)$ be an H-matrix and (B_k, C_k, E_k), $k = 1(1)\ \alpha$ multisplitting of A with $\text{diag}(A) = D = \text{diag}(B_k)$, $k = 1(1)\ \alpha$, satisfy following:

(a) $\langle A \rangle = \langle B_k \rangle - \langle C_k \rangle = |D| - |B|$, $k = 1(1)\alpha$

(b) $|G(x) - G(y)| \le P\,|x - y|$, $0 \le P \in L(R^n)$, $x, y \in R^n$

(c) $\rho(A^{-1}\, P) < 1$

(d) $\omega \in (0, 2 / (1 + \rho))$, $\rho := \rho\,(A^{-1}\, P)$

Then, the method (3.12) converges to the unique solution x^* for starting $x^0 \in R^n$.

PROOF: We only need to prove that

$$\left[\ \omega\, \rho(A^{-1}\, P) + |1 - \omega|\ \right] < 1$$

$$|\varsigma| \le \left[\ |1 - \omega|\, I + \omega \left\{ \sum_{k=1}^{\alpha} E_k\, |B^{-1}|\ |C_k| + \sum_{k=1}^{\alpha} E_k\, |B^{-1}|\, P \right\}\right]$$

$$= \left[\ |1 - \omega|\, I + \omega \sum_{k=1}^{\alpha} E_k\, |B^{-1}|\ \left\{\ |C_k| + P \right\}\right]$$

since, $\langle C_k \rangle = \langle B_k \rangle - \langle A \rangle$

$$\le \left[\ |1 - \omega|\, I + \omega \sum_{k=1}^{\alpha} E_k\, |B^{-1}|\ \left\{\ \langle B_k \rangle - \langle A \rangle + P \right\}\right]$$

By lemma (2.8(b)) $|B^{-1}| \le \langle B_k \rangle^{-1}$ inequality holds

$$\leq \left[|1 - \omega| I + \omega \sum_{k=1}^{\alpha} E_k \langle B_k \rangle^{-1} \left\{ \langle B_k \rangle - \langle A \rangle + P \right\} \right]$$

$$= \left[|1 - \omega| I + \omega I - \omega \sum_{k=1}^{\alpha} E_k \langle B_k \rangle^{-1} \langle A \rangle \left\{ I - \langle A \rangle^{-1} P \right\} \right]$$

By lemma (2.4) and (2.6(c)),
$\langle A \rangle = |D| (I - J)$, $\langle A \rangle^{-1} = (I - J)^{-1} |D|^{-1}$ and $|D|^{-1} \leq \langle B_k \rangle^{-1}$, inequality holds respectively

$$\leq \left[\left\{ |1 - \omega| + \omega \right\} I - \omega \sum_{k=1}^{\alpha} E_k |D|^{-1} |D| (I - J) (I - (I - J))^{-1} |D|^{-1} P \right]$$

$$\leq |1 - \omega| + \omega - \omega (I - J) + \omega (I - J) (I - J)^{-1} |D|^{-1} P$$

$$= |1 - \omega| + \omega - \omega + \omega J + \omega |D|^{-1} P$$

By lemma (2.4) $J = |D|^{-1} |B|$

$$\leq |1 - \omega| I + \omega |D|^{-1} |B| + \omega |D|^{-1} P$$

$$= |1 - \omega| I + \omega (|D|^{-1} (|B| + P))$$

$$\rho(|\varsigma|) \leq (|1 - \omega| I + \omega \rho(|D|^{-1} (|B| + P))$$
$$\leq |1 - \omega| I + \omega \rho < 1$$
$$\rho(|\varsigma|) \leq 1$$

THEOREM 4.2: Let $A \in L(R^n)$ be an H-matrix and $(D - L_k, U_k, E_k)$, $k = 1(1)\alpha$ multisplitting of A with $\text{diag}(A) = D = \text{diag}(B_k)$, $k = 1(1)\alpha$, satisfy the following:

(a) $A = |D| - |L_k| - |U_k| = |D| - (|L_k| + |U_k|) = |D| - |B|$, $k = 1(1) \infty$

(b) $|G(x) - G(y)| \leq P |x - y|$, $0 \leq P \in L(R^n)$, $x, y \in R^n$

(c) $\rho(A^{-1} P) < 1$

(d) $\omega \in (0, 2/(1 + \rho))$, $\rho := \rho(A^{-1} P)$, $0 \leq r > \omega$

Then, the relaxed multisplitting AOR method (3.4) converges to the unique solution x^* for starting point $x^0 \in R^n$.

PROOF: We only need to prove $\rho(\varsigma) < 1$

$$(\varsigma) \leq \Sigma E_k (D - r L_k)^{-1} \left[|1 - \omega| |D| + (\omega - r) |L_k| + \omega |U_k| + \omega P \right]$$

Then the $D - r L_k$, $k = 1(1)\alpha$, are H-matrices, Hence the inequality,

$$|D - r L_k| \leq \langle D - r L_k \rangle^{-1} = (|D - r L_k|)^{-1}, \quad k = 1(1)\alpha,$$

$$\leq \sum_{k=1}^{\alpha} E_k (|D| - r |L_k|)^{-1} \left[|1 - \omega| |D| + (\omega - r) |L_k| + \omega |U_k| + \omega P \right]$$

$$\leq \sum_{k=1}^{\alpha} E_k \, (|D| - r\,|L_k|)^{-1} \left[(|1 - \omega| - 1)|D| + (|D| - r\,|L_k|) + \omega(|L_k| + |U_k|) + \omega\, P \right]$$

By theorem (4.2(a)) $|L_k| + |U_k| = |B_k|$

$$\leq \sum_{k=1}^{\alpha} E_k \, (|D| - r\,|L_k|)^{-1} \left[(|1 - \omega| - 1)|D| + (|D| - r\,|L_k| + \omega\,|B|) + \omega \right]$$

$$= I + \sum_{k=1}^{\alpha} E_k \, (|D| - r\,|L_k|)^{-1} \left[(|1 - \omega| - 1)|D| + \omega\,(|B| + \omega P) \right]$$

$$\leq I + \sum_{k=1}^{\alpha} E_k \, (|D| - r\,|L_k|)^{-1} |D| \left[(|1 - \omega| - 1)I + \omega\,|D|^{-1}(|B| + P) \right] \quad (4.3)$$

Let e defined by the $e^T = (1, 1, \ldots, 1)^T \in R^n$. Since J is nonnegative ($J \geq 0$) the matrix $J + \varepsilon\, e\, e^T$ has only positive entries and is irreducible for any $\varepsilon > 0$. By the theorem of Perron and Frobenius [8], for any $\varepsilon > 0$ there is vector $x_\varepsilon > 0$ such that $(J + \varepsilon\, e\, e^T)\, x_\varepsilon = \rho_\varepsilon\, x_\varepsilon$, where $\rho_\varepsilon = \rho(J + \varepsilon\, e\, e^T)$ have ρ_ε we have $\rho_\varepsilon < 1$. By continuity of the spectral radius we also obtain

$$|1 - \omega| + \omega\, \rho_\varepsilon < 1 \quad (4.4)$$

If $\varepsilon > 0$ is sufficiently small. In this case we get from (4.3)

$$|L| \leq I + \sum_{k=1}^{\alpha} E_k \, (|D| - r\,|L_k|)^{-1} |D| \left[(|1 - \omega| - 1)\, I + \omega \left(|D|^{-1}((|B| + \rho) + \varepsilon\, e\, e^T) \right) \right]$$

and then, multiplying by x_ε,

$$|L|\, x_\varepsilon \leq x \sum_{k=1}^{\alpha} + E_k \, (|D| - r\,|L_k|)^{-1} |D| \left[(|1 - \omega| - 1)\, I + \omega \left(|D|^{-1}((|B| + \rho) + \varepsilon\, e\, e^T) \right) \right] x_\varepsilon \quad (4.5)$$

$$(|D| - r\, L_k) \leq |D|$$

$$(|D| - r\, L_k)^{-1}\, |D| \geq I$$

Then from (4.5) we get

$$|L|\, x_\varepsilon \leq x_\varepsilon + \left[(|1 - \omega| - 1)\, x_\varepsilon + \omega\, \rho_\varepsilon\, x_\varepsilon \right]$$

$$= x_\varepsilon + |1 - \omega|\, x_\varepsilon + \omega\, \rho_\varepsilon\, x_\varepsilon$$

$$|L|\, x_\varepsilon \leq (|1 - \omega| - \omega\, \rho_\varepsilon)\, x_\varepsilon < x_\varepsilon$$

Therefore, $(|L|\, x_\varepsilon)_i \,/\, (x_\varepsilon)_i < 1$, for i = 1(1)n and Ex. 2 in [p 47] guarantee $\rho(|L|\, x) < 1$.

Hence, under the assumption of the theorem, this completes the proof theorem (4.2).

5 Monotone convergence:

Obviously we have

$$B^{-1} A = I - B^{-1} C_k \tag{5.1}$$

and from

$$H = \sum_{k=1}^{n} E_k B^{-1} C_k, \quad Q = \sum_{k=1}^{n} E_k B^{-1}$$

we get,

$$Q A = I - H \tag{5.2}$$

The fact that the spectral radius of H, is less than one, $\rho(H) < 1$. existence of A^{-1} imply that Q s invertible and satisfying

$$Q^{-1} = A (I - H)^{-1}$$

$$A = Q^{-1} - Q^{-1} H \tag{5.3}$$

As $(Q^{-1})^{-1} = Q \geq 0$, $(Q^{-1})^{-1} Q^{-1} H = H \geq 0$. We can easily see that the single splitting (5.3), induced by the multisplitting $A = B_k - C_k$, for $k = 1(1)\alpha$ weak regular. Here we should note that the splitting (5.3) is not usual regular.

The above discussion shows that the multisplitting iteration met (3.2) is convergent under the condition, which we have given. Now, we first prove that the iteration (3.2) has monotone convergence property under the condition.

THEOREM 5.4: Let $A \in L(R^{-1})$, $A^{-1} \geq 0$, (B_k, C_k, E_k), for $k = 1(1)\alpha$ by multisplitting of A with $A = B_k - C_k$, for $k = 1(1)\alpha$ being weak regular.

$G : R^n \rightarrow R^n$ is isotone mapping, i.e. $x \geq y$, $G(x) \geq G(y)$. Let $x^0, y^0 \in 1$

$x^0 \leq y^0$ be initial points satisfying

$$A x^0 \leq G(x^0) \tag{5.4}$$

$$A y^0 \geq G(y^0) \tag{5.5}$$

Then, the sequence $\{x^m\}$ and $\{y^m\}$ generated by the multisplitting iteration

$$x^{m+1} = H x^m + Q G(x^m) \tag{5.6}$$

$$y^{m+1} = H y^m + Q G(y^m) \tag{5.7}$$

converges to the unique solution x^* of the system of non-linear equation (1.1) respectively and satisfying

$$x^0 \leq x^1 \leq, \ldots, \leq x^m \leq x^{m+1} \leq x^* \leq y^{m+1} \leq y^m \leq, \ldots, \leq y^1 \leq y^0 \tag{5.10}$$

PROOF: We have known that the sequences $\{x^m\}$ and $\{y^m\}$ converges to the unique solution x^* of the system of non-linear equations (1.1) we now prove that (5.10) is true. We only need to test,

(1) For all m, there hold $x^m \leq x^{m+1}$, $y^m \geq y^{m+1}$

(2) For all m, there hold $x^m \leq x^* \leq y^m$

Here we will use induction to prove (i) for $m = 0$ the (5.8) becomes:

$x^1 = H x^0 + Q G(x^0)$

subtract x^0 to both sides,

$$x^1 - x^0 = \sum_{k=1}^{n} E_k B^{-1} C_k x^0 - x^0 + \Sigma E_k B^{-1} G(x^0)$$

$$= \sum_{k=1}^{n} E_k B^{-1} (C_k - B_k) x^0 + \Sigma E_k B^{-1} G(x^0)$$

$$= \sum_{k=1}^{n} E_k B^{-1} (-Ax^0 + G(x^0)) \geq 0$$

$x^1 - x^0 \geq 0$ By (5.6) and $Q \geq 0$.

$x^0 \leq x^1$

Now, assume that the result (1) is

$x^{n+1} = H x^n + Q G(x^n)$

$x^n = H + Q G(x^{n-1})$

we get,

$x^{n+1} - x^n = H (x^n - x^{n-1}) + Q \left(G(x^n) - G(x^{n-1})\right) \geq 0$

By (5.5) and (1), $H \geq 0$ and $Q \geq 0$, We get,

$x^{n+1} - x^n \geq 0$

$x^n \leq x^{n+1}$

we can conclude that $x^n \leq x^{n+1}$ hold for all m.

Similarly we can test the mono tonicity of the sequence $\{y^m\}$.

For (2) we need to prove (i) x^*, y^* are the unique solution of the system non-linear equation (1.1)

(ii) $x^* = y^*$

for (i) $x^* \in R^n$ is the unique solution of (1.1) and G is continuous satisfy.

$$x^* = H x^* + Q G(x^*) \tag{5.11}$$

Similarly we get,

$$y^* = H y^* + Q G(y^*) \tag{5.12}$$

for (ii)

Subtract (5.12) from (5.11), we obtain,

$$|x^* - y^*| \leq |H| |x^* - y^*| + |Q| |G(x^*) - G(y^*)| \tag{5.13}$$

By theorem [3.18(b)], we may write Eq. No. (5.13) in this form

$$|x^* - y^*| \leq \left(|H| + |Q|\rho\right) |x^* - y^*| \tag{5.14}$$

under the condition of theorem (3.18) we already proven that

$\rho(|H| + |Q| \rho) < 1$, then (5.14) give us

$|x^* - y^*| = 0 \quad \Rightarrow \quad x^* - y^* = 0$

$x^* - y^*$

Now Subtract (5.11) from (5.8) we get,

$x^{m+1} - x^{*} = H(x^{m} - x^{*}) + Q\left(G(x^{m}) - G(x^{*})\right) \leq 0$

Here we use $H \geq 0$, $Q \geq 0$, by (5.5) and the induction assumption. There identically hold $x^{m} \leq x^{*}$, $m = 0, 1, \ldots$

Similarly $y^{m} \geq y^{*}$, $m = 0, 1, \ldots$ can be obtained.

THEOREM 5.15: Let condition of Theorem (5.4) be satisfied and $Q^{-1} H \geq 0$,

Then,

$A x^{m} \leq G(x^{m})$, $m = 0, 1, \ldots$

$A y^{m} \leq G(y^{m})$, $m = 0, 1, \ldots$

The sequences $\{x^{m}\}$ and $\{y^{m}\}$ are defined by (5.8) and (5.9) respectively.

PROOF: This inequality (5.17) can be easily obtained by making the induction. We only need to prove:

$$\begin{aligned} A x^{m+1} &= (Q^{-1} - Q^{-1} H) x^{m+1} \\ &= Q^{-1} (I - H) x^{m+1} \\ &= Q^{-1} (H x^{m} + Q G(x^{m}) - H x^{m+1}) \\ &= Q^{-1} (H x^{m} - x^{m+1}) + G(x^{m}) \leq G(x^{m}) \end{aligned}$$

By (5.16) and $x^{m} - x^{m+1} \leq 0$

$A x^{m+1} \leq G(x^{m})$

$x^{m} \leq x^{m+1}$

and G is isotone

$G(x^{m}) \leq G(x^{m+1})$

Then (5.19) can be written as,

$A x^{m+1} \leq G(x^{m+1})$.

Similarly, we can also get the inequality (5.18) immediately $A y^{m+1} \geq G(y^{m})$ And

$A y^{m+1} \geq G(y^{m+1})$.

5. Comparison theorem:

In this section, by applying the comparability of the multisplitting iterative sequence, we describe the different convergence rate of the multisplitting iterative methods constructed by different multisplittings.

THEOREM 6.1: Let $A \in L(R^{n})$, $A^{-1} \geq 0$,

$$A = B_{k_1} - C_{k_1} = B_{k_2} - C_{k_2} \qquad (6.2)$$

For $k = 1(1)\infty$, be two different weak, regular splittings of A and E_{k}, be diagonal weighting matrix. Assume that the sequence $\{x^{m}\}$ and $\{y^{m}\}$ are defined by:

$$x^{m+1} = H_1 x^m + Q_1 G(x^m),\ H_1 = \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} C_{k_1},\ Q_1 = \sum_{k=1}^{\alpha} E_k B_{k_2}^{-1} \tag{6.3}$$

$$y^{m+1} = H_2 x^m + Q_2 G(x^m),\ H_2 = \sum_{k=1}^{\alpha} E_k B_{k_2}^{-1} C_{k_2},\ Q_2 = \sum_{k=1}^{\alpha} E_k B_{k_2}^{-1} \tag{6.4}$$

Where

$$H_i = \sum_{k=1}^{\alpha} E_k B_{k_i}^{-1} C_{k_1},\ Q_i = \sum_{k=1}^{\alpha} E_k B_{k_i}^{-1},\ i = 1, 2, \ldots \tag{6.5}$$

And take

$$x^0 = y^0, \text{ Then if } Q_1^{-1} H_1 \geq 0 \tag{6.6}$$

$$B_{k_1}^{-1} \geq B_{k_1}^{-1}, \text{ for } k = 1(1)\alpha \tag{6.7}$$

There hold

$$x^m \geq y^m,\ m = 0, 1, \ldots, \quad \text{as } A x^0 \leq G(x^m) \tag{6.8}$$

$$x^m \leq y^m,\ m = 0, 1, \ldots, \quad \text{as } A x^0 \leq G(x^m)$$

PROOF: (6.8) and (6.9) can be proved similarly as sum in theorem we only need to verify (6.8). As

$E_k \geq 0$ for $k = 1(1)\alpha$

By (6.7) we get,

$Q_1 = Q_2 \geq 0$

From (6.8)

$A x^0 \leq G(x^0)$

We know that there exists a non-negative vector $U_0 \geq 0$ such that:

$$A x^0 + U_0 = G(x^0)$$

From (6.2) we also obtain

$$B_{k_i} x^0 + U_0 = C_{k_i} x^0 + G(x^0),\ I = 1, 2,\ \ k = 1(1)\alpha$$

From (6.3) we have,

$$x^1 = H_1 x^0 + Q_1 G(x^0)$$

$$= \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} C_{k_1} x + \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} G(x^0)$$

$$= \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} \left(C_{k_1} x + G(x^0)\right) = \sum_{k=1}^{\alpha} E_k B_{K_1}^{-1} (B_{k_1} x^0 + U_0)$$

$$= \sum_{k=1}^{\alpha} E_k x^0 + \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} U_0$$

$$= x^0 + Q_1 U_0 \geq x^0 + Q_2 U_0$$

$$= \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} (B_{k_2} x^0 + U_0) = \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} \left(C_{k_2} x^0 + G(x^0)\right)$$

$$= H_2 y^0 + Q_2 G(x^0)$$

$$= y^1.$$

which shows that (6.8) is true for m = 0. Now, we assume that (6.8) hold m = n, that is $x^m \geq y^m$, m = 0(0)n. By theorem (5.15), we know $A x^n \leq G(x^n)$. Thus there exists $U_n \geq 0$, such that:

$$A x^n + U_n = G(x^n) \tag{6.9}$$

From (6.2) we also get,

$$B_{k_i} x^n + U_n = C_{k_i} x^n + G(x^n), \; i = 1, 2, \; n = 1(1)\alpha \tag{6.10}$$

From (6.3), we have

$$x^{n+1} = H_1 x^n + Q_1 G(x^n)$$

$$= \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} C_{k_1} x^n + \Sigma E_k B_{k_1}^{-1} G(x^n)$$

$$= \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} \left(C_{k_1} x^n + G(x^n)\right) = \Sigma E_k B_{k_1}^{-1} (B_{k_1} x^n + U_n)$$

$$= x^n + \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} U_n$$

$$= x^n + Q_1 U_n \geq x^n + Q_2 U_n$$

$$= \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} (B_{k_2} x^n + U_n) = \sum_{k=1}^{\alpha} E_k B_{k_1}^{-1} \left(C_{k_2} x^n + G(x^n)\right)$$

$$= H_2 y^n + Q_2 G(x^n)$$

$$= y^{n+1}$$

Thus by induction for all value of m, $x^m \geq y^m$ inequality hold.

THEOREM 6.15: Let $A \in L(R^n)$, $A^{-1} \geq 0$, $A - B_k - C_k$ for $k = 1(1)\alpha$, regular splitting of A and E_k for $k = 1(1)\alpha$, be weighting matrix

$A = \underline{B} - \underline{C} = \overline{B} - \overline{C}$ be two Regular splittings of A, satisfying

$\underline{B} \leq B_k \leq \overline{B}$ for $k = 1(1)\alpha$.

Assume that the sequences $\{x^m\}$, $\{y^m\}$ and $\{z^m\}$ are defined by:

$$x^{m+1} = H x^m + Q G(x^m), \qquad m = 0, 1, \ldots$$

$$y^{m+1} = B^{-1} C y^m + B G(x^m), \quad m = 0, 1, \ldots$$

$$z^{m+1} = \overline{B}^{-1} \overline{C} z^m + \overline{B} G(x^m), \quad m = 0, 1, \ldots$$

and take $x^0 = y^0 = z^0$. Then

If $y^m \geq x^m \geq z^m$, m = 0, 1, ..., we have $A\, x^k \leq G(x^0)$

If $y^m \leq x^m \leq z^m$, m = 0, 1, ..., we have $A\, x^k \geq G(x^0)$

We can get the proof of this theorem by taking

$A = B_{k_1} - C_{k_1} = \underline{B} - \underline{C}$, $k = 1(1)\alpha$

$A = B_{k_1} - C_{k_1} = \overline{B} - \overline{C}$ in theorem (6.1).

7 Applications:

In this section, we apply the theory of § 6 to several multisplitting proposed in [2] for this purpose, we assume that $A \in L(R^n)$ an M-matrix and not loss generality, we take diag(A) = I, consider following multisplittings:

$A = (I - L_k) - (U + L - L_k)$, $k = 1(1)\ \alpha$

$A = (I - L_k)(I - U_k) - (U + L + L_k U_k - L_k - U_k)$, $k = 1(1)\ \alpha$

Where L, U are strictly Lower, Upper triangular matrices respectively, and $0 \leq L_k \leq L$, $0 \leq U_k \leq U$, $k = 1(1)\ \alpha$

obviously, we see that

$(I - L) \leq (I - L_k) \leq I$, $k = 1(1)\alpha$

$(I - L)(I - U) \leq (I - L_k)(I - U_k) \leq I$, $k = 1(1)\ \alpha$

and the corresponding single Jacobi, Gauss Seidel as well as symmetric Seidel splitting of A are all regular, we know that iteration methods:

$$x_{MGS}^{m+1} = L_{MGS}\, x_{MGS}^{m} + G_{MGS}(x_{MGS}^{m}),\ x_{MSGS}^{m+1} = L_{MSGS}\, x_{MSGS}^{m} + G_{MSGS}(x_{MSGS}^{m})$$

$$x_{J}^{m+1} = L_J\, x_J^m + G_J(x_J^m),\ x_{GS}^{m+1} = L_{GS}\, x_{GS}^{m} + G_{GS}(x_{GS}^{m})$$

$x_{SGS}^{m+1} = L_{SGS}\, x_{SGS}^{m} + G_{SGS}(x_{SGS}^{m})$, with

c

$$L_{MGS} = \Sigma\, E_k (I - L_k)^{-1} (U + L - L_k),\ G_{MGS}(x_{MGS}) = \Sigma\, E_k (I - L_k)^{-1} G(x)$$

$$L_{MGS} = \Sigma\, E_k (I - U_k)^{-1} (I - L_k)^{-1} (U + L + L_k U_k - L_k - U_k),$$

$$G_{MSGS}(x_{MSGS}) = \Sigma\, E_k\, (I - U_k)^{-1} (I - L_k)^{-1} G(x).$$

$L_J = L + U$, $G_J(x_J) = G(x)$

$$L_{GS} = (I - L)^{-1} U,\ G_{GS}(x_{GS}) = (I - L)^{-1} G(x)$$

$$L_{SGS} = (I - U)^{-1} (I - L)^{-1} L\, U,\ G_{SGS}(x_{SGS}) = (I - U)^{-1} (I - L)^{-1} G(x)$$

Corresponding to the above splitting satisfy the following relations,

(i) when $A x^0 \leq G(x^0)$, n = MGS, MSGS, J, GS, SGS, there hold

$$x^m_{GS} \geq x^m_{MGS} \geq x^m_J, \; x^m_{SGS} \geq x^m_{MSGS} \geq x^m_J, \; m = 0, 1, \ldots$$

(ii) when $A x^0 \geq G(x^0)$, n = MGS, MSGS, J, GS, SGS, there hold

$$x^m \leq x^m_{MGS} \leq x^m_J, \; x^m_{SGS} \leq x^m_{MSGS} \leq x^m_J, \; m = 0, 1, \ldots$$

when $x^0_{MGS} = x^0_{MSGS} = x^0_J = x^0_{GS} = x^0_{SGS}$.

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