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CENTRAL DECOMPOSITIONS OF FINITE P-GROUPS WITH CENTER OF ORDER P.

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Abstract

If G is a finite p-group with Z (G) or order p, then (according to Hall-senior James theorem) either G is extra-special or Z_2 (G) is abelian or G is centrally decomposable with G =AYB, where A is extra-special and Z_2 (B) is abelian. Here the central decompositions of these three cases separately is discussed.

Keywords: The central decompositions of P -Groups.

Introduction

The concept of central decompositions of groups is a specialization of the notion of generalized direct decompositions which was introduced by B. H. Neumann and H. Neumann [10]. In general the central factors of a central decomposition of a given group are not determined upto isomorphism. The main object of this study is to draw attention to the question of uniqueness of such decompositions and to find out to what extent Remak-Krull-Schimd type of results can be obtained under suitable conditions.

The study of central decompositions of groups was initiated by Tang [11,12] from a more general approach including the case of infinite groups. A quick glance through literature on the given problem shows that the structure theorems involving central decompositions are about p-groups (cf: structure of extra-special p-groups and P. Hall theorem (see [5] Theorems 5,4,9 & 5,5,2),structure of p-groups with cyclic Frattini subgroups [4], Hall –Senior-James theorem [6,8] e.t.c).

According to Hall-Senior-James theorem* if G is a finite P-group with center of order p, then either (1) G is an extra-special or (2) Z_2 (G) is abelian or (3) G is centrally decomposable with G = AYB, where A is an extra-special and Z_2 (B) is abelian.

These kind of results provide the foundation for the deeper consideration of the results of the type Remak-Krull-Schimdt theorem for a special class of groups (as, in general, for ordinary finite p-groups such type of results are not accessible); and that is why we consider the classification of central decompositions of centrally decomposable finite P-groups with center of order P.

To study such decompositions, naturally, one would like to answer the following two question:

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*Hall-Senior [6] (Theorem 4.2) proved this result for 2-groups of nil potency class 3. James, in his Ph. D thesis [8] (Theorem 2.7), proved that this result is true for an arbitrary prime groups and Pointed out that the restriction on nil potency class does not exist.
(The detail version of this talk appear in [1],[2] and [3])

(i) What are the possible central decompositions of a given group G, namely, what is the exact (formal)number of such decompositions.?

(ii) Are all of them (pairwise) isomorphic ?

Thus , if G is a finite-group with Z (G) of order p then (according to Hall-Senior James theorem) either G is extra-special or Z_2 (G) is abelain or G is centrally decomposable with G = AYB, where A is extra-special and Z_2 (B) is abelian. Here these three possibilities are separately discussed. The detailed versions of these results appear in [1], [2]and [3] respectively.

§1. A finite set $\{U_1,\ldots,U_r\}$ of proper subgroups of a group G is said to be a *generalized direct decomposition* (in short g.d.d) if

(i) G. is generated by $U_1,..,U_r$,i.e $G < U_1 ,.., U_r >$,

(ii) For $I \neq j$, U_i and U_j commute element wise ; i.e $[U_i \ U_j]= \{e\}$; and we write G= g.d.d $\{ U_i ,U_j \}$.

Such decomposition of G is aid to be a *central decomposition* (ink short c.d) if $Z (G) \leq U_i$ for all i =1,….r, In that case each U_i is called central *factor* of G and we write $G = U_i Y\ldots \ Y U_r$

If G possesses such decomposition , then G is said to be *centrally decomposable*, other wise G is *centrally indecomposable.*

Note that a c.d $G = U_i Y_{\ldots Y} U_r$ induces a direct decomposition of the factor group G/Z .i.e $G/Z (G) = U_i /Z (U_i)$ x…x $U_r / Z (U_r)$.

A c.d $G = U_i Y \ldots \ Y u_r$ of G is said to be *unrefinable central* decomposition (in short u.c.d), if each U_i is centrally indecomposable.

Two u.c.d, say, $G = U_i Y \ldots \ Y u_r$ and $G = V_1 \ Y \ldots \ Y V_s$ of a group G are said to be isomorphic if r =s and for each i =1,…,r there exists j in {1,…,s} such that U_i is i*somorphic* to U_j.

Let U and V be two subgroups of a group G Then U is said to be *centrally isomorphic* to V , if there exists an isomorphism P:U $\rightarrow$ V such that (up) u^{-1} belongs to Z (G).

Note that if U is centrally isomorphic to V, then UZ (G) = VZ(G).

Proposition 1.1 (Remaak-Krull-Schmidt theorem (See, for example, [9], pp 120)). If a group G has principal series, then any two direct decomposition of G with directly indecomposable factors are centrally isomorphic.

If G is centrally decomposable, then, in general, the central factors o f a u.c.d of G are not determined upto isomorphism and hence two u.c.d of G are not necessarily isomorphic. Even more, tang [12] proved that two u.c.d of a group may not are equal number of central factors, however, an immediate consequence of Remak-Krull-Schmidt theorem, proves that

Proposition 1.2 If $G = U_i Y \ldots Y U_r$ and $G = V_1 Y \ldots Y V_s$ are two u.c.d of G such that each $U_i / Z(U_i)$ and each $V_j / Z(V_j)$ is directly indecomposable, then $r = s$.

Again, if $G = U_i Y \ldots Y U_r$ and $G = V_1 Y \ldots Y V_s$ are two u.c.d of G such that $r = s$, even then they need not to be isomorphic. For example if

$D = < a,b , a^{2n-1} = e = b^2 b^{-1} ab = a^{-1} >$

$Q = < c,d; c^{2n-1} = e, d^2 = c^{2n-1}, d^{-1} cd = c^{-1} >$

are dihedral and Generalized Quaternion, respectively, 2-groups of maximal class and order 2^n, where $n \geq 4$, then $G = DYD$ and $G = QYQ$ are two u.c.d of G are not isomorphic.

However, in the following Lemma, if two c.d of a group G have a common central factor, then the other central factors are determined by the common central factor.

Lemma 1.3 Let $G = AY B$ and $G = A Y C$ be two c.d of G then $B = C_G(A) = C$.

§2. Unless other wise stated, all groups under consideration are finite P-groups with center of order P.

A non abelian group G is aid to be an extra-special, if the derived subgroup $G' = Z(G) = \phi(G)$ – the fatting subgroup of G, has order P. It is a well known [5] theorem 5.5.1) that a non-abelian group G of order P^3 is extra-special and is isomorphic to one to the following groups:

$M = < x, y, z; [x,y] = z, x^p = y^p = z^p = e = [x, z] = [y, z] >$

$N = < u, v, u p^2 = v^p = e, v^{-1} uv = u^{-1} >$ for $p \geq 3$; and

$D_3 = < x,y , x^{2^2} = y^2 = e , y^{-1} xy = x^{-1} >$

$Q_3 = < u,v , u^{2^2} = e , u^2 v^2 = v^1 uv = u^{-1} >$ for $p = 2$.

Further a nonabelian group of order p^3 is centrally indecomposable and an extra-special group G has a u.c.d into nonabelian subgroups each of order p^3 and therefore, G has order p^{2r+1}, $r \geq 2$ (see ,for example, [1]).

Theorem 2.1 [1] (Theorem 6). Let G be extra-special group of order p^{2r+1}, $r \geq 2$. Then G has exactly $(1/r!) . p^{2r-1} p^{2r-1} \ldots\ldots (p^4 - 1)/(p^2 - 1)^{r-1}$ u.c.d.

Proposition 2.2. ([5] Theorem 5.5.2). Let G be extra-special of order p^{2r+1} $r \geq 2$. If G has exponent P then all of the u.c.d of G are , pair wise, isomorphic. In fact each of the u.c.d of G is isomorphic to $M^r = M \underbrace{Y .. Y} M$. If G has not exponent pl then each of the u.c.d of G is

isomorphic to either $N^t M^{r-t}$ for $p \geq 3$ or $D_3^t Q_3^{r-t}$, for $p = 2$, where $1 \leq t \leq r$. Moreover $N^t M^{r-t}$ is isomorphic, to $N M^{r-1}$ if $t \geq 1$ and M^r is not isomorphic to $N M^{r-1}$ for $p \geq 3$; and $D_3^t Q_3^{r-t} \cong D_3^t Q_3^{r-1}$, if $t \geq 1$ and $Q_3^r \cong D_3^t Q_3^{r-1}$ for $p = 2$. In particular, $N^2 \cong NM$ and $M^2 \cong NM$; and $D^2_3 \cong Q^2_3$, and $D_3 Q_3 \cong D^2_3$.

Lemma 2.3 Let G be a group with $Z_2(G)$ of order p^2 . Then G is centrally indecomposable and $K = C_G(Z_2(G))$ is a maximal subgroup of G.

Now lets consider central decompositions of those groups G such that $G = U_1 Y \ldots\ldots Y\ U_r$, where each $Z_2(U_1)$ has order p^2, is a u.c.d of G so that $Z_2(G)$ = g.d.d $\{ Z_2(U_i); I = 1,\ldots,r\}$ is abelian of order p^{r+1}, $r \geq 2$. because other wise there exist p – groups with Z (G) of order p, Z_2 (G) abelian of order p^3 and Z_2 (G) = g.d.d {A,B} , where both A and B has order p^2 , yet G is centrally indecomposable.

Example 2.4 Let

$$G = UT(4,p) = < \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_4 & a_5 \\ 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; a_i \varepsilon\ z^p > \text{then}\quad Z(G) = < \begin{bmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > \text{has}$$

order p; and Z_2 (G) = < x_1, x_2, x_3 > where

$$x_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that $Z_2(G) / Z(G) = < x_1 Z(G), x_2 Z(G), >$, where $x_1 Z(G) \cap x_2 Z(G) = Z(G)$; and therefore, $Z_2(G)$ = g. d.d $\{< x_1, x_3 >, < x_2, x_3 > \}$, where each $< x_1, x_3 >$ and x_2, x_3 has order P^2.

Suppose G = A Y B, where A,B, both are nonabelian subgroups. Then either G/Z (G) = A/Z(A) x B/z(B) and Z(G/Z(G)) = Z(A/Z((A)) XZ(B/Z(B)). Sine $|G/Z(G)| = P$ either $|A/Z(A)| = P^3$ and $|B/Z(B)| = P^2$ or vise versa. If $|B/Z(B)) = Z(G/Z(B))| = P^2$,then B is extra-special of order P^3 and B/Z(B) =Z(B/Z(B)) =Z(G/Z(G)) has order p^2. In that case Z(A/Z(A)) is trivial, which is not possible. Hence G is centrally indecomposable.

Lemma 2.5 ([2] Lemma 2). Let G = U_1 Y … Y U_r be a.u.c.d of G with each $Z_2(U_i)$ of order 2. If G = G = V_1 Y … Y V_s is a .u.c.d of G with each $Z_2(V_j)$ of order P^2, that

i) r =s and G = V_1 Y .. Y V_s is a.u.c.d of G.
ii) For I = 1, …, r, $U_i Z_2(G) = V_j Z_2)G)$, where $1 \leq j \leq s$.
iii) $M_{i,} = C_u(Z_2(U_i))$ is a maximal subgroup in Vj for some $1 \leq j \leq s$.

Lemma 2.6 ([2] Lemma 2). Let G = U_1 Y … Y U_r be a u.c.d of G with each $Z_2(U_i)$ of order 2 p^2. I = 1,…, r. Then G has exactly p(2)u.c.d; and if Z_2 (G) is elementary abelian (as it is for $p \geq 3$, see [7], Theorem 3.7.7.), then all u.c.d. of G are pair wise isomorphic.

If p = 2, then Z_2 (G) is not necessarily elementary abelian,. Examples of each groups are the 2-groups of maximal nil potency class, namely, D-the Dihedral group, Q- the Generalized Quaternions and S-the Semidihedral group; each of order 2^n, $n \leq 4$: Theorem all u.c.d of G are not necessarily isomorphic).

Lemma 2.7 ([2] Theorem 3). Let $G = U_1 Y \dots Y U_r$ be a.u.c.d of G with each $U_i \varepsilon$ {S.D.Q }. Then each u.c.d. of G is isomorphic to either $S^{r-t} D^t$, $0 \le t \le r$ or QD_{r-1} Moreover

$$S^s D^d Q^q \cong \begin{cases} D^{q+d-1} Q \text{ if } s = 0 \text{ and } q \text{ odd,} \\ S^s D^{q+d} \text{ otherwise.} \end{cases}$$

§3. In this section we consider central decompositions of those groups G such that $G = U_1 Y \dots\dots Y U_r = Y U_r Y v_1 Y \dots\dots Y v_s$ is a.u.c.d of G, where each $Z_2(U_i)$ has order P^2, and each V_j is extra-special of order P^3; and, therefore, $Z_2(G)$ = g.d.d. $\{ Z_2(U_i), V_j \}$ is upon abelian. For detail, see [3].

Lemma 3.1 Let $G = U_1 Y \dots Y U_r = Y v_1 Y \dots Y v_s$ be a u.c.d of G with each $Z_2(U_i)$ of order P^2, and each V_j – non abelian of order P^3; if $G = A_1 Y \dots\dots A_m Y B_1 \dots\dots Y B_n$ is a.c.d of G such that each $Z_2(A_k)$ has order P^2 and each B_j is non abelian of order P^3, then

i) $r = m$ and $s = n$; and, therefore, $G = A_1 Y \dots A_m Y B_1 \dots Y B_n$ is a u.c.d of G.
ii) For each $i = 1, \dots, r$, $U_i Z_2(G) = A_k Z_2(G)$ for $k \varepsilon \{ 1, \dots, m \}$.
iii) $M_i = C_{U_i}(Z_2(U_i))$ is a maximal subgroup in A_k for $1 \le k \le m$.

Lemma 3.2 Let $r, s \ge 1$ $G = U_1 Y \dots Y U_r Y v_1 Y \dots Y v_s$ be a.u.c.d of G, where each $Z_2(U_i)_r$ has order P^2, and each V_j is nonabelian of order P^3. Then G has exactly

◀ $(1/s1), P^{2+2rs\,s(s-1)} .(p^{2s}-1) \dots (p^4-1)/(p^2-1)^{s-1}$

◀ u.c.d.

Corollary 3.3. Let $G = U_1 Y \dots Y U_r Y v_1 Y \dots Y v_s$ be a.u.c.d of G, as in Theorem 3.2 Then, if p is odd, then all u.c.d of G are pairwise isomorphic. In particular, if $Z_2(G)$ is of exonent p, then each u.c.d of G is isomorphic t $G = U_1 Y \dots Y U_r Y M^s$: $Z_2(G)$ has not exponent P, then each u.c.d of G is isomorphic $G = U_1 Y \dots Y U_r Y NM.^{s-1}$

If p=2, then, of course, $Z_2(G)$ has not exponent p; and all of u.c.d of G are not necessarily isomorphic. Examples of such groups are those 2 – groups G such that $G = U_1 Y \dots Y U_r Y v_1 Y \dots Y v_s$ a.c.d of G with each $U_i \varepsilon \{ S.D. Q \}$ and each $V_j \varepsilon \{ D_3, Q_3 \}$. In this case we have the following result:

Theorem 3.5. $G = U_1 Y \dots Y U_r Y v_1 Y \dots Y v_s$ be a.u.c.d of G, is a.u.c of G such that each $U_i \varepsilon \{S.D.Q\}$ and each $V_j \varepsilon \{D_3, Q_3\}$. Then each u.c.d of G isomorphic to either $S^{r-t} D^t Q_3^s$, $0 \le t \le r$ or $D^r Q_3^s \cong Q^r D_3^s$, or $D^r D_3^s \; Q^r Q_3^s$. Moreover

$$S^s D^d Q_3^q \cong \begin{cases} Q^d D_q^3, \text{ if } s = 0 \\ S^s D^d D_3^q \cong S^s Q^d Q_3^q \cong S^s D^d D_3^q \text{ otherwise.} \end{cases}$$

References

[1]. Abbasi, G.Q.- Extra-special q-groups and central decompositions. The Punjab Univ.Jour. of Maths. 1984-85. (XVII-XVIII) 63-68.

[2]. Abassi, G.Q. Central decompositions of finite p – groups with abelian second center and the center of order p. Vestnik MGU Ser.1, Mech. Math 1985 N0.2. 75-78.

[3]. Absssi, G.Q.- Central decompositions of finite p-groups with nonabelian second center and center of order p. The Pun.Univ.J of Maths. (Lahore0 29(1996)., 1-10.

[4]. Berger, T; Kovacs, L.G; Newman, M,F.- Groups of prime power order with cyclic Frattini subgroups. Cambridge Phil. Soc. A 1980 (83) 13-18.

[5]. Gorenstien, D.-Finte Groups. Chelsea Publishing Co. New York, 1980.

[6]. Hall, M.Jr; Senior, J.K.- The groups of order 2^n, $n \leq 6$. The Macmillan Co.N.Y 1964.

[7]. Hupert, B.- Endliche Gruppen I. Springer Verlag New York, 1967.

[8]. James, R.K.- The groups of order p^6, $p \geq 3$. Ph.d thesis, Univ. of Sydney, 1969.

[9]. Kurosh, A.G.- The groups of Groups (Vol.II). Chelsea Publishing Co. New York, 19[illegible].

[10]. Neumann, B.H; Neumann, H.- A remark on generalized free products. Journal of London Math. Soc. 1950 (25) 201-204.

[11]. Tang, C.Y.- On uniqueness of generalized direct decompositions. Pacific Journal of Math. 1967 (23) 171-182.

[12]. Tang, C.Y. On uniqueness of central products of groups. Pacific journal of Math. 19[illegible] (33) 749-761.