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AMBIGUOUS NUMBERS UNDER THE ACTION OF THE GROUP $\langle x, y: x^2=y^4=1\rangle$

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Abstract

We discuss here some basic properties of certain real quadratic irrational numbers α , called ambiguous numbers, under the action of $G= \langle x, y: X^2=y^4=1 \rangle$by using coset diagrams. We show that if α is totally positive then $y(\alpha)$, $y^2(\alpha)$, $y^3(\alpha)$ are totally negative. Also the linear fractional transformation x transforms totally positive number to a totally negative number and vice-versa, and an ambiguous number into an ambiguous number. We show also that if $\alpha$is an ambiguous number then one of $y(\alpha)$, $y^2(\alpha)$,$y^4(\alpha)$ *is ambiguous and the other two are totally negative.*. Finally, we show that $\alpha G \subseteq Q^*(\sqrt{n})$, where $Q^*(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : a, c \in Z, b = \frac{a^2-n}{c}, (a, b, c) = 1 \right\}$ and the action of G on the projective line over $Q^*(\sqrt{n})$ is always intransitive.

Keyword: **Modular Group, Coset Diagram, Real Quadratic Irrational Numbers, Ambiguous Numbers**

1. Introduction

We are interested in linear-fractional transformations x, y satisfying the relations $x^2 = y = 1$, with a view to studying an action of the group (x, y) on real quadratic fields. If $y : z \rightarrow \frac{az+b}{cz+d}$ is to act on all real quadratic fields then a,b,c,d must be rational numbers, and can be taken to be integers. Thus $\frac{(a+b)^2}{ad-bc}$ is rational. But if $y : z \rightarrow \frac{(az+b)}{az-d}$is of order m one must have $\frac{(a+b)^2}{ad-bc} = \omega + \omega^{-1} + 2$, where ω is a primitive m-th root of unity. Now $\omega + \omega^{-1}$ is rational, for a primitive m–th root ω, only if $m = 1,2,3,4,6$. So these are the only possible orders of y. The group $\langle x, y \rangle$ is the modular group PSL (2, Z) and its action on the real quadratic irrational numbers have been discussed in detail in [2], [3], and [5]. For some fixed non-square positive integer n, an element

$\alpha = \frac{a+\sqrt{n}}{c}$ and its conjugate may $\overline{\alpha} = \frac{a-\sqrt{n}}{c}$ have different signs. If such is the case then we shall call such α an ambiguous number. If α and $\overline{\alpha}$ are both positive (negative), then we shall call α a totally positive (negative) number.

There real quadratic irrational numbers α have been classified in [2] into the orbit of α under PSL (2,Z). In [3], it has been shown that classifying α is almost the same as classifying indefinite binary quadratic forms and that under a certain condition an indefinite binary quadratic form s is equivalent to $-s$.

There are two projections from the projective line over the real quadratic field to the projective line over a finite field. In [5], certain necessary and sufficient conditions are determined for the existence of two orbits of M: one containing α along with $\frac{1}{\alpha}$ and the other containing α along with $\frac{1}{\overline{\alpha}}$.

In this short note we are interested in the group $\langle x, y \rangle$ for $m = 4$, that is, $G = \langle x, y: X^2 = y^4 = 1 \rangle$, acting on the real quadratic irrational numbers, where $x(z) = \frac{-1}{-2z}$ and

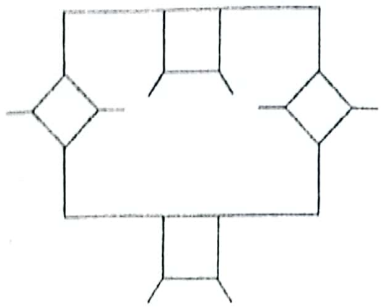
$y(z) = \frac{-1}{2(z+1)}$ are linear - fractional transformations. An action of G on real quadratic irrational numbers has been considered in [4]. It has been shown that the set of ambiguous numbers is finite and that part of the coset diagram containing these numbers forms a single closed path, which is the only closed path in the orbit of α.

We discuss here some basic properties of the real quadratic irrational numbers α under the action of G by using coset diagrams. We show that if α is totally positive then $y(\alpha)$, $y^2(\alpha)$, $y^3(\alpha)$ are totally negative. Also the linear-fractional transformation x transforms totally positive number to a totally negative number and vice-verse; and an ambiguous number into an ambiguous number. We show also that if α is an ambiguous number then one of $y(\alpha)$, $y^2(\alpha)$, $y^3(\alpha)$ is ambiguous and the other two are totally negative. Finally, we show that $a\ G \subseteq Q^*(\sqrt{n})$, and the action of G on the projective line over $Q^*(\sqrt{n}:) = \{\frac{a+\sqrt{n}}{c}: a, c \in Z, b = \frac{a^2-n}{c}, (a, b, c) = 1\}$ is always intransitive.

2. Coset Diagrams

We use coset diagrams for the group G and study its action on the projective line over real quadratic fields. The coset diagrams for the group G are defined as follows. The four cycles of the transformation y are denoted by four edges of a square permuted anti-clockwise by y and the two vertices which are interchanged by x are joined by an edge. Fixed points of x and y, if they exist, are denoted by heavy dots.

For instance, in the case of G acting on the projective line over the field $Q^*(\sqrt{5})$ a fragment of the coset diagram will be



Consider a coset diagram for the action of G on the projective line over a real quadratic field. If $k \neq 0, \frac{-1}{2}, -1, \infty$ is one of a square in the coset diagram, then k and $x(k)$ are end points of an edge. Since $x(k) = \frac{-1}{2k}$, one of k and $x(k)$ is positive and the other is negative. Since $y: \rightarrow z \frac{-1}{2(z)+1)}$,

(i) if $z < -1$, then $y(z) > 0$,

(ii) if $z > 0$, then $\frac{-1}{2} < y(z) < 0$,

(iii) if $\frac{-1}{2} < z < 0$, then $-1 < y(z) < \frac{-1}{2}$,

(iv) if $-1 < z < \frac{-1}{2}$, then $y(z) < -1$.

That is, if vertices k, $y(k)$, $y^2(k)$, $y^3(k)$ of a square are not $0, -1, \frac{-1}{2}, \infty$ then just one of them lies in each of the intervals: $(-\infty, -1)$, $(-1, \frac{-1}{2})$, $(\frac{-1}{2}, 0)$ and $(0, \infty)$.

Thus, in particular, of the vertices k, $y(k)$, $y^2(k)$, $y^3(k)$ one is positive and three are negative.

3. Main Results

It is easy to see that a linear-fractional transformation $z \rightarrow \frac{\quad}{\quad}$ has order n if $A^n = \lambda I$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Here we prove some easy but useful results for later use.

Leema 3.1

(i) $A^2 = \lambda I$ if and only if the trace of A, tr (A), is 0.

(ii) $A^4 = \lambda I$ if and only if $\{ \text{tr } (A)\}^2 = 2 \det (A)$.

Proof (i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

If $A^2 = \lambda I$ then $a^2 + bc = bc + d^2$ implies that $a^2 = d^2$ and $ab + bd = ac + cd = 0$. Hence $a + d = 0$ or the trace of A is equal to 0.

(ii) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}, \text{ if } A^4 = \lambda I.$$

Then tr $(A^2) = 0$, and so, $a^2 + 2bc + d^2 = 0$ or $(a + d)^2 = 2 (ad - bc)$ or $\{ \text{tr } (A)\}^2 = 2 \det (A)$.

We will need the following lemmas from [1]. We reproduce them here without proofs.

Lemma 3.2

An $\alpha = \frac{(a+\sqrt{n})}{c}, \in Q^*(\sqrt{n})$, is a totally positive number if and only if either $a,b,c > 0$ or $a,b,c < 0$.

Lemma 3.3

An $\alpha = \frac{(\alpha+\sqrt{n})}{c}, \in Q^*(\sqrt{n})$ is a totally negative number if and only if either $\alpha < 0$ and $b, c > 0$ or $\alpha > 0$ and $b, c < 0$.

Lemma 3.4

An $\alpha = \frac{(\alpha+\sqrt{n})}{c}, \in Q^*(\sqrt{n})$ is an ambiguous number if and only if either $b > 0$ and $c < 0$ or $b < 0$ and $c > 0$.

Theorem 3.5

Let $\alpha = \frac{(\alpha+\sqrt{n})}{c}$ be a totally positive irrational quadratic number then the real quadratic irrational numbers $y(\alpha)$, $y^2(\alpha)$, $y^3(\alpha)$ are totally negative.

Proof

Since $y(z) = \frac{-1}{2(z+1)}$, therefore,

$$Y(\alpha) = \frac{-1}{2(a+1)} = \frac{-a-c+\sqrt{n}}{2(2a+b+c)} = \frac{\alpha_1+\sqrt{n}}{c_1}$$

where $a_1 = -a - c$, $c_1 = 2(2a + b + c)$ and $b_1 = \frac{a_1^2 - n}{c_1} = \frac{c}{2}$. Similarly we tabulate the values of a,b,c in the following fashion.

a	a	b	c
$y(a)$	$-a-c$	$\frac{c}{2}$	$2(2a+b+c)$
$y^2(a)$	$-3a-2b-c$	$2a+b+c$	$4a+4b+c$
$y^3(a)$	$-a-2b$	$\frac{4a+4b+c}{2}$	$2b$

Since α is totally positive, therefore, either $a,b,c > 0$ or $a, b, c < 0$. If $a, b, c > 0$ then from the above table we can easily see that $y(a)$, $y^2(a)$, $y^3(a)$ are totally negative; and on the other hand if $a, b, c < 0$ then $y(a), y^2(a), y^3(a)$ are totally negative.

Theorem 3.6

Let $\alpha = \frac{(\sqrt{n})}{c}, \in Q^*(\sqrt{n})$. Then x transforms a totally positive number α into a totally negative number and a totally

negative number α into a totally positive number.

Proof

Let $\alpha = \frac{\alpha + (\sqrt{n})}{c}$ be a totally positive number.

Then $x(\alpha) = \frac{-1}{2a} = \frac{-(\alpha + (\sqrt{n}))}{2b} = \frac{\alpha_1 + (\sqrt{n})}{c_1}$

where $a_1 = -a$, $c_1 = 2b$ and $= b_1 \frac{a_1^2 - n}{c_1} = \frac{c}{2}$.

Since α is a totally positive number, either $a,b,c > 0$ or $a,b,c < 0$. Then $a_1 < 0$ and $b_1,c_1 > 0$,and if $a,b, c < 0$ then $a_1 >$ and $b_1,c < 0$. Hence $x(a)$ is totally negative in either case.

If a is a totally negative number then either $a > 0$ and $b,c < 0$ or $a < 0$ and $b,c > 0$. If $a > 0$ and $b, c < 0$ then $a_1,b_1,c_1 < 0$. If $a < 0$ and $b,c > 0$ then $a_1,b_1,c_1 > 0$. Hence $x(a)$ is a totally positive number.

Theorem 3.7

Let $a = \frac{\alpha + (\sqrt{n})}{c} \in Q^*(\sqrt{n})$ be an ambiguous number. If $x(z) = \frac{1}{nz}$, where $m \in Z$ then $x(a)$ is a totally positive number.

Proof

As $x(z) = \frac{1}{Mz}$ so $x(a) = \frac{1}{ma} = \frac{-a + (\sqrt{n})}{-bm}$

where $\frac{(-a)^2 - n}{-bm} = \frac{-c}{m}$, since a is an ambiguous number, therefore $bc < 0$. Thus $\frac{(-c)}{m}(-bm) = bc < 0$ implies that $x(a)$ is an ambiguous number.

If, in theorem 3.7, we put $m = -2$ we obtain the standard linear-fractional transformation $x(z) = \frac{-1}{2z}$ which is one of the generators of the group G.

Theorem 3.8

Let $\alpha = \frac{\alpha + (\sqrt{n})}{c}$ be an ambiguous number. Then one of $y(\alpha)$, $y^2(\alpha)$, $y^3(\alpha)$ is ambiguous and the other two are totally negative.

Proof If α is one of the four vertices of a square in the coset diagram then because of the argument given at the end of section 2, one of $\alpha, y(\alpha)$, $y^2(\alpha)$, $y^3(\alpha)$ is positive and the other three are negative. Let α, β, γ, δ be the vertices of a square in the coset diagram. First we show that if α, β, γ, δ are the vertices of a square then α, β, γ, δ are also vertices of a square in a coset diagram.

If $\alpha = \frac{\alpha + (\sqrt{n})}{c}$ where $b = \frac{\alpha^2 - n}{c}$

then $\bar{\alpha} = \frac{\alpha - (\sqrt{n})}{c}$ $\frac{\alpha + (\sqrt{n})}{-c}$ $\frac{\alpha^2 - n}{-c}$ $= -b$. We tabulate the information as follows.

α	α	b	c
$y(\alpha)$	$-\alpha - c$	$\frac{-c}{2}$	$2(2\alpha + b + c)$
$y^2(\alpha)$	$-3\alpha - 2b - c$	$2\alpha + b + c$	$4\alpha + 4b + c$
$y^3(\alpha)$	$-\alpha\ 2b$	$\frac{4\alpha + 4b + c}{2}$	$.2b$
$\bar{\alpha}$	$-\alpha$	$-b$	$-c - c$
$y(\bar{\alpha})$	$\alpha + c$	$\frac{-c}{2}$	$-2b\ (2\alpha + b + c)$
$y^2(\bar{\alpha})$	$3\alpha + 2b + c$	$-(2\alpha + b + c)$	$-(4\alpha + 4b + c)$
$y^3(\bar{\alpha})$	$\alpha\ 2b$	$(4\alpha + 4b + c)$	$-2b$

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Hence $\overline{y}(a) = y(\overline{a})$, $\overline{y}^2(a) = y^2(\overline{a})$, $\overline{y}^3(a) = y^3(\overline{a})$.

Hence if α. $Y(\alpha)$, $y^2(a)$, $y^3(\alpha)$ are vertices of a square then $\overline{a}$, $\overline{y} = y(a)$, $\overline{y}^2(a)$, $\overline{y}^3(a)$ are also vertices of a square in a coset diagram. Since one of α and $x(\alpha)$ is positive, therefore, without any loss of generality, we consider the ambiguous number α as positive. So the only possibilities for the signs of these numbers are:

$\alpha, \beta, \gamma, \delta$	$\overline{\alpha}$,	$\overline{\beta}$,	$\overline{\gamma}$,	$\overline{\delta}$
+ - - -	-	+	-	-
	-	-	+	-
	-	-	-	+

Hence one of $y(\alpha)$, $y^2(a)$, $y^3(a)$ is ambiguous and the other two are totally negative.

Theorem 3.9

Let $\alpha = \frac{(\alpha+\sqrt{n})}{2c} \in Q^*(\sqrt{n})$. Then $\alpha G \subseteq Q^*(\sqrt{n})$.

Proof we can easily tabulate the following information.

α	α	b	$2c$
$x(\alpha)$	$-\alpha$	c	$2b$
$y(\alpha)$	$-\alpha - 2c$	c	$2(2\alpha + b + 2c)$
$y^2(\alpha)$	$-3\alpha - 2b - 2c$	$2\alpha+b+2c$	$2(2\alpha+2b+c)$
$y^3(\alpha)$	$-\alpha - 2b$	$2\alpha + 2b + c$	$2b$
$yx(\alpha)$	$\alpha+2c$	$2\alpha+b+2c$	$2c$
$y^2x(\alpha)$	$3\alpha+2b+2c$	$2\alpha+2b+c$	$2(2\alpha+b+2c)$
$y^3x(\alpha)$	$\alpha+2b$	b	$2(2\alpha+b+2c)$

From the above table we see that all the numbers in αG are of the form $\frac{\alpha+\sqrt{n}}{2m}$, where $m \in Z$. Hence they belong to $Q^*(\sqrt{n})$ and so $\alpha G \subseteq Q^*(\sqrt{n})$.

Theorem 3.10

The action of G on the projective line over $Q^*(\sqrt{n})$ is intransitive.

Proof:

$\alpha = \frac{(\alpha+\sqrt{n})}{2c} \in Q^*(\sqrt{n})$. Then we tabulate the action of x, y on α as follows.

α	α	b	$2c$
$x(\alpha)$	$-\alpha$	c	$2b$
$y(\alpha)$	$-\alpha - 2\alpha$	c	$2(2\alpha + b + 2c)$
$y^2(\alpha)$	$-3\alpha - 2b - 2c$	$2\alpha+b+2c$	$2(2\alpha+2b+c)$
$y^3(\alpha)$	$-\alpha - 2b$	$2\alpha + 2b + c$	$2b$

Hence all the elements in αG are of the form $\frac{\alpha+\sqrt{n}}{2m}$, where $m \in Z$. Particularly, all the ambiguous numbers in $a G$ are of the form $\frac{\alpha+\sqrt{n}}{2m}$.

Let $\beta = \frac{\alpha+\sqrt{n}}{2c+1}$ be an element of $Q^*(\sqrt{n})$. Then the application of x, y on β yield the following information.

β	α	b	$2c + 1$
$x(\beta)$	$-\alpha$	$\frac{c+1}{2}$	$2b$
$y(\beta)$	$-\alpha-2c-1$	$\frac{c+1}{2}2$	$(2\alpha+b+2c+1)$
$y^2(\beta)$	$-3\alpha-2b-2c-1$	$2\alpha+b+2c+1$	$4\alpha+4b+2c+1$
$y^3(\beta)$	$-\alpha-2b$	$2\alpha + 2b + \frac{c+1}{2}$	$2b$

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Hence βG contain only those elements of $Q^*(\sqrt{n})$, which are of the form $\frac{\alpha+\sqrt{n}}{2c+1}$, where $c \in Z$. Since n is a non-square positive integer, therefore, n must be greater than 1. Let $k = 1+\sqrt{n}$ then $\alpha = 1$, $b = 1-n$, $c = 1$. Therefore, $k \in Q^*(\sqrt{n}$ k $k = 1 - n < 0$ implies that k is an ambiguous number.

Here two cases arise, we therefore deal with them separately in the following fashion.

a) Let n be even, that is, $n = 2m$. If we take $k = \frac{(\sqrt{n}}{2}$, *so* $a = 0$, $b = \frac{0-n}{2} =$ $m, c = 2$. Hence $k \in Q^*(\sqrt{n})$. As $bc = -n < 0$, k is an ambiguous number.

b) Suppose now that n is odd, that is, $n = 2m + 1$. Let $k = \frac{1+\sqrt{n}}{2}$ $\alpha = 1$, $b = \frac{1-n}{2} = -m, c = 2$.

Hence $k \in Q^*(\sqrt{n})$. As $bc = -2m < 0$, so k is an ambiguous number. Hence for every non-square n there exist ambiguous numbers of the form $\frac{a+(\sqrt{n})}{2c}$ and $\frac{\alpha+(\sqrt{n})}{2c+1}$ in $Q^*(\sqrt{n})$, but they do not lie in the same orbit. As $Q^*(\sqrt{n})$ has more than two orbits, therefoe, the action of G on the projective line over $Q^*(\sqrt{n})$ is intransitive.

4. Conclusion

We conclude this paper with the following observations. The situation here is very different from the one in [3]. If α is of the form $\alpha + \frac{(\sqrt{n})}{2c}$, where $c \in Z$ then we can find a closed path in the coset diagram of αG in which all the ambiguous numbers are of the form $\alpha + \frac{(\sqrt{n})}{2m}$, where $m \in Z$, and they all belong to $Q^*(\sqrt{n})$. But if α is not of the form $\frac{\alpha+(\sqrt{n})}{2m}$, where $k \in Z$, then we can find a closed path in the coset diagram of α G in which not all the ambiguous numbers belong to $Q^*(\sqrt{n})$.

For some value of n, there is more than one orbits containing numbers of discriminant n. For instance, the closed paths associated with the transformations xy^3 xy^2 and xy^2 xy^3 contain numbers of discrininant 7.

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