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ON THE TRIANGLE GROUP Δ (2,3,9)

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Abstract

The infinite triangle group $\Delta(2,3,9) = \langle x,y: x^2=y^3=(xy)^9=1 \rangle$ has a fundamental domain consisting of two copies of a hyperbolic triangle with angles $\frac{\pi}{2}$ $\frac{\pi}{3}$ $\frac{\pi}{9}$

Let *a, b, c* be the vertices of the hyperbolic triangle and R_i represent hyperbolic-reflection in hyperbolic sides ac, cb and ab. Let $x = R_9 R_3$ and $y = R_2 R_9$, so that $yx = R_2 R_9 R_9 R_3 = R_2 R_3$. Then, $R_9 R_3$ is an anticlockwise hyperbolic-rotation or π about *a*, $R_2 R_9$ is an anticlockwise hyperbolic-rotation of $\frac{2\pi}{3}$ about *b*, and $R_2 R_3$ is an anticlockwise hyperbolic-rotation of 2π about *c*. Hence, $x^2 = y^3 = (xy)^9 = 1$ which is $\Delta(2, 3, 9)$. The space that $\Delta(2,3,9)$ acts on, is hyperbolic.

It is known that conjugacy classes of actions of *PGL* (2, Z) on $F_q \cup \{\infty\}$ can be represented by coset diagrams *D* (θ, q) where $\theta \in F_q$ and q is a power of a prime p. In this paper we have obtained conditions on θ and q which guarantee only those coset diagrams which depict homomorphic images of the infinite triangle group $\Delta(2, 3, 9)$ in *PGL* (2, q).

Keywords: Triangle groups Coset diagrams Fragments, Projective general linear groups, Modular group, and Projective lines over Galois fields.

Introduction

It is well known that the modular group *PSL* (2,Z) is generated by the two linear-fractional transformations $x: z \to -\frac{1}{z}$

$y: z \to \frac{z-1}{z}$, which satisfy the relations

(1.1) $\quad x^2 = y^3 = 1$

The linear-fractional transformation $t: z \to \frac{1}{z}$ inverts x, y, that is ,

$t^2 = (xt)^2 = (yt)^2 = 1$

and so extends the group *PSL* (2,Z). the extended modular group *PGL* (2,Z) is then generated by x,y and t and its defining relations are:

(1.2) $\quad x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1$

Let *q* be a power of a prime *p*. then the group *PGL* (2,*q*) has its customary meaning, as the group of all transformations $z \to \frac{az+b}{cz+d}$ where *a,b,c, d* are in F_q and *ad-bc* $\neq 0$, while the group *PSL* (2, *q*) is its subgroup consisting of all those linear-fractional transformations $z \to \frac{az+b}{cz+d}$ where *ad-bc* is

a non-zero square in F_q.

Let *PL* (F_q) denote the projective line over F_q , that is , $F_q \cup \{\infty\}$. Every element of *PGL* (2, *q*) gives a permutation on the points of *PL* (F_q), and so *PGL* (2,*q*) is a subgroup of the symmetric group S_{q+1}. as the elements of *PSL* (2,*q*) give only even permutations, it is therefore a subgroup of the alternating group A_{q+1}.

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Triangle groups and their significance is well explained in [5]. Triangle groups are represented by

(1.3) $\Delta(l,m,n) = \langle x,y: x^l = y^m = (xy)^n = 1 \rangle$

where $l, m, n > 1$.

It is well-known that $\Delta(l, m, n)$ is isomorphic to a subgroup of $PSL(2,C)$, that x is of order l, y is of order m and xy is of order n. if

(1.4) $s(\Delta(l, m, n)) = \frac{1}{l} + \frac{1}{m} + \frac{1}{n}$

then $\Delta(l, m, n)$ contains the fundamental group of a positive genus orientable surface as a subgroup of finite index whenever $s(\Delta(l, m, n)) \leq 1$; in particular $\Delta(l, m, n)$ is infinite. Furthermore, if $s(\Delta(l, m, n)) < 1$, the genus of the surface is greater than 1and consequently $\Delta(l, m, n)$ contains a free group of rank 2 [1].

The triangle group $\Delta(2, 3, n)$ are specially important for being homomorphic images of $PSL(2,Z)$. These triangle groups are infinite if and only if $n \geq 6$. The finite groups $\Delta(2, 3, n)$, $n \leq 5$ are known and they are: trivial, S_3 A_4, S_4, and A_5 for n = 1,2,3,4, and 5 respectively. When $n = 6$, then the group $\Delta(2, 3, 6)$ is an extension by the cyclic group C_6 of a free abelian group of rank 2. $\Delta(2, 3, 6)$ is infinite but soluble. Its commutator subgroup is a free Abelian group on two generators, and the associated factor-commutator group is cyclic of order 6. A study of this group can be found in [6]. For n = 7, the triangle group $\Delta(2, 3, n)$ becomes a Hurwitz group which is extensively studied in [2, 7, 10]. $\Delta(2, 3, 7)$ is not just infinite and insouable, but in fact SQ-universal, that is, every countable group occurs as a subgroup of $\Delta(l, m, n)$. No information about the triangle group $\Delta(2, 3, 8)$ is known. We are therefore interested in triangle groups $\Delta(2, 3, 9)$.

Triangle group $\Delta(2, 3, 9)$ is studied in [9] but in a different context. They have a fundamental domain consisting of two copies of a hyperbolic triangle with angles $\frac{\pi}{2}$ $\frac{\pi}{3}$ $\frac{\pi}{9}$

Let a,b,c be the vertices of the hyperbolic triangle and R_i represent hyperbolic-reflection in hyberbolic sides ac, cd and ab. Let $x = R_9 R_3$ and $y = R_2R_9$, so that $yx = R_2 R_9 R_9 R_3 = R_2 R_3$.Then R_9 R_3 is an anticlockwise hyperbolic-rotation of π about a, R_2 R_9 is an anticlockwise hyperbolic-rotation of $\frac{2\pi}{9}$ about c.

Hence, $x^2 = y^2 = (xy)^9 = 1$ which is $\Delta(2, 3, 9)$. The space that $\Delta(2, 3, 9)$ acts on, is hyperbolic. In this paper we study homomorphic images of $\Delta(2, 3, 9)$ in $PGL(2, q)$, that is , as subgroups of i_{q+1} or i_{q+1} and the related coset diagrams.

Coset Diagrams

A coset diagram corresponding to a subgroup H of finite indixz in a finitely-generated group G, is a directed, edge-coloured graph, whose vertices are the (right) coset of H in G and whose edges are defined as follows: we take a specific set of generators for each generator x and each vertex Hg, draw an edge of colure C_x from Hg to Hgx. This is a generalization of the Cayley colour graph corresponding to a (finite) presentation for G. Every transitive permutation representation of a group corresponds naturally to one on the cosets of a subgroup. Any transitive permutation representation of a finitely-

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generated group on a projective line over finite fields $PL\ (F_q)$ can be depicted graphically by a coset diagram. Indeed, this can be done even without specific knowledge of the associated subgroup. One needs only the permutations induced by specific generators on the points of $PL\ (F_q)$. We use coset diagrams for he actions of the extended modular group $PGL\ (2, Z)$ on $PL\ (F_q)$. they are defined as follows. The three cycles of y are represented by small triangles whose vertices are permuted counter clockwise by y. the vertices which are interchanged by x, are represented by edges and action of t is represented by the reflection about the vertical line of symmetry. The fixed points of x and y, if exist, are denoted by heavy dots; and to make the diagram less complicated the loops representing the fixed points of x are omitted. In this case (of a connected diagram), the vertices are identifiable with the right cosets in $PGL\ (2,Z)$ of the stabilizer N of any given point of $PL\ (F_q)$, so that x or y joins the coset N_{gx} or N_{gy} (for each g ∈ i (2,Z)).

For instance , consider the action of PGL (2,Z) on PL (F_{19}) which is $x\ (z) = \frac{-1}{z}$,

$y\ (z) = \frac{z-1}{z}$,$t\ (z) = \frac{1}{z}$ where $z \in PL\ (F_{19})$.

we can calculate the permutation representation of x,y and t as follows:

x: (0,∞) (1,18) (2,9) (3,6) (4,14) (5,15) (7,8) (10,17) (11,12) (13,16).

y: (0,∞,1) (2,10,18) (3,7,9) (4,15,6) (5,16,14) (8) (12) (13,17,11), and

t: ((0,∞)(1) (2,10) (3,13) (4,5) (6,16) (7,11) (8,12) (9,17) (14,15) (18).

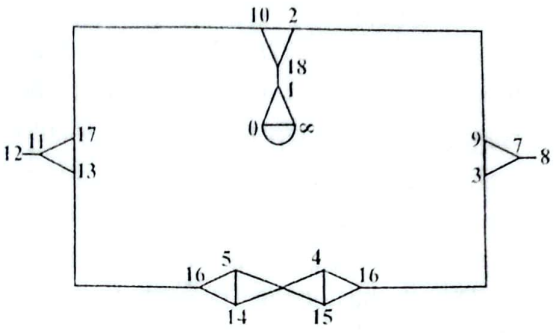


Fig - 1

Coset diagrams may be used to provide diagrammatic interpretations of several aspects of combinatorial group theory, such as the Reidemeister-Schreier procedure, as well as a proof of the Ree-Singerman theorem (on the cycle structures of generating-permutations of a transitive group). They can be used also as an equivalent to the Ablianized form of the Reidemeister-Schreier process. The same sort of method is also useful for the construction of infinite families of finite quotients of a given finitely-presented group. Use of coset diagrams to find torsion-free subgroups of certain finitely-presented group has been instrumental in the construction of small volume hyperbolic 3-orbifolds and other hyperbolic 3-manifloes with interesting properties. They are also applied to the construction of arc-transitive graphs and maximal automorphism groups of Riemann surfaces [3].

A homomorphism $\alpha : PGL\ (2, Z) \rightarrow PGL\ (2,q)$ gives rise to an action of PLG (2, I) on $PL\ (F_q)$. If none of the generators x and y for $PSL\ (2,Z)$ lies in the kernal of α, so that images $x\alpha = \overline{x}$ and $y\alpha = \overline{y}$ are of orders 2 and 3 respectively, then α is said to be a non-degenerate homomorphism. This implies that the kernel contains the normal

closure of this, and hence we get a map from $PGL(2,Z)$, Which is essentially $<(xy)^9>$ $\Delta(2, 3, 9)$, to $PGL(2, q)$. Two such homomorphisms α and β are said to be conjugate if $\beta = \alpha\rho$ for some inner auto orphism ρ of $PGL(2, q)$. If $x^2 = y^3 = t^2 = 1$ and $x\alpha = \overline{x}$, $y\alpha = y$, and $t\alpha = \overline{t}$, then orders of $\overline{x}$, $\overline{t}$ is 2 and of $\overline{y}$ is 3. In [8] it has been proved that the cojugacy classes of no-degenerate homomorphisms of $PGL(2, Z)$ into $PGL(2, q)$ correspond in a one-to-one fashion with the conjugacy classes of no-trivial elements of $PGL(2, q)$, under a correspondence which assigns to the non-degenerate homomorphism α the class containing the element $(x\ y)\ \alpha$. This of course, means that we can actually parametrise the conjugacy Classes of non-degenerate homomorphisms $\alpha : PGL(2, Z) \rightarrow PGL(2, q)$, except for a few uninteresting ones, by the elements of F_q. That is, we can in fact parametrise the actions of $PGL(2, Z)$ on $PL(F_p)$.

If α is any such non-degenerate homomorphism, and X, Y and T non-singular 2 x 2 matrices with entries from F_q representing the images $x\alpha = \overline{x}$, $y\alpha = \overline{y}$, and $t\alpha = \overline{t}$, so that X, Y and T satisfy, up to a scalar multiplier, the relations (1.2), then letting $\theta = \frac{r^2}{\Delta}$, where r and Δ are the trace and the determinant of XY, we associate the parameter θ with α. We shall take $\overline{x}$, $\overline{y}$, $\overline{t}$ to the represented by the matrices $X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}$,

$Y = \begin{bmatrix} d & kf \\ f & -d-1 \end{bmatrix}$, and $T + \begin{bmatrix} o & -k \\ 1 & o \end{bmatrix}$

Which satisfy $X^2 = Y^3 = T^2 = \lambda I$, where λ is some non-zero scalar. The action of $PGL(2,Z)$ on $PL(F_q)$ evolves $PGL(2, q)$ and the corresponding coset diagram yields a permutation representation of $PGL(2, q)$. In [8] a mechanism ahs been developed to find a unique coset diagram $D(\theta,q)$ corresponding to each $\theta \in F_q$. It is unique in the sense that the actions corresponding to the same conjugacy class will produce the same coset diagram, except for the labeling of the vertices.

In this paper are interested in only those conjugacy classes of non-degenerate homomorphism α which evolve the pairs of linear-fractional transformations x,y which satisfy the relation $x^2 = y^3 = (xy) = 1$.

Theorem 1. for each zero of $f(z) = z^3 - 6z^2 + 9z - 1$ in F_p, where $f(z) \in Z[z]$ and p is a primer number such that $p \equiv \pm 1 \pmod 9$, there exists a conjugacy class of non-degenerate homomorphisms α such that $\alpha(PGL(2, Z)) = <\overline{x}, \overline{y} : \overline{x}^2 = \overline{y}^3 = (\overline{x}\ \overline{y})^9 = 1>$.

Proof. Since $p \equiv \pm 1 \pmod 9$, there are three distinct traces r_1, r_2, r_3 of elements of the group $SL(2, p)$, and thus there are three conjugacy classes of non-degenerate homomorphisms α from $\Delta(2, 3, 9)$ into $PGL(2, p)$ [4]. Now suppose A is any element of $SL(2, p)$ which has trace $r = r_1, r_2$, or r_3. As A is conjugate in $GL(2, p)^2$ to a matrix B of the form $\begin{bmatrix} p & 0 \\ 0 & p^{-1} \end{bmatrix}$, where p is supposed to be a primitive 9^{th} root of unity in F_{p2} thus $r = tr(A) = tr(B) = p + p^{-1}$. Next, $r^2 = (\rho + \rho^{-1})^2 = \rho^2 + \rho^{-2} + 2$, so $r^2 - 2 = \rho^2 + \rho^{-2}$, which is the trace of B^2; and $r^3 = (\rho + \rho^{-1})^3 = \rho^3 + 3\rho^{-3} + 3\rho^{-1} + \rho^{-3}$, so $r^3 - 3r = \rho^3 + \rho^{-3}$, which is the trace of B^3. Now since $\rho^9 = 1$ and $\rho \neq 1$, we have $\rho^8 = \rho^{-1}$, $\rho^7 = \rho^{-2}$, $\rho^6 = \rho^{-3}$, $\rho^5 = \rho^{-4}$.

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That is , $r^2 = (\rho + \rho^{-1}) + (\rho^2 + \rho^{-2}) + (\rho^3 + \rho^{-3}) + (\rho^4 + \rho^{-4}) + 1 = 0$ which implies that $r^4 + r^3 - 3r^2 - 2r + 1 = 0$ or $r(r^2 - 2) = -r^4 + 3r^2 - 1$. Squaring both sides we get $r^8 - 7r^6 + 15r^4 - 10r^2 + 1 = 0$. Since tr $(B) = r$ and det $(B) = \Delta = 1$, put $\theta = 1$, put $\theta = r^2$ in the equation to obtain $i^4 - 7\theta^3 + 15\theta^2 - 10\theta + 1 = 0$ or $(\theta - 1)(\theta^3 - 6\theta^2 + 9\theta - 1) = 0$. Since $\rho \neq 1$, therefore $\theta - 1 \neq 0$, and so $\overline{xy}$ has order 9 if and only if $f(\theta) = \theta^3 - 6\theta^2 + 9\theta - 1 = 0$.

The coset diagram $D(\theta p)$, where θ is a zero of $\overline{f}(z) = z^3 - 6z^2 + 9z - 1$ in F_9 and $p \equiv \pm 1 \pmod 9$, will be such that each vertex in the diagram will be fixed by $(x\ y)^9$. The three zeros of $f(z) = z^3 - 6z^2 + 9z - 1$ lies in F_9 if $p = 9m \pm 1$, where $m \in Z$. Corresponding to each zero of $f(\theta) = 0$ in F_p, we can construct a coset diagram using the technique described in [7]. For instance $f(z) = 0$ has three roots 5, 9, 11 in F_p,. If we choose $\theta = 5$, then the linear-fractional transformations $\overline{x}$, $\underline{y}$, $\underline{t}$, will be respectively:

(0,16) (1,2) (3,15) (4,11) (5,6) (7,10) (8,17) (9,18) (12,14) (13, ∞), (0,17,15) (1,12,2) (3,14,13) (4,6,16) (5,10, ∞) (7) (8) (9,11,18), and (0, ∞) (1,18) (2,9) (3,6) (4,14) (5,15) (7,8) (10,17) (11,12) (13,16).

Therefore , the coset diagram will be D (5,19) in which each vertex is fixed by $(x\ \overline{y})^9 = 1$. Thus it will be a diagram depicting the homomorphci images of Δ (2,3,9).

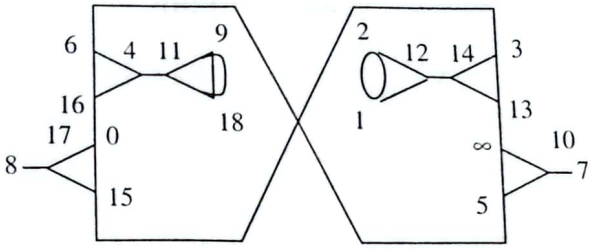


Fig - 2

In the following we give a list of triplets $\underline{x}$, $\underline{y}$, $\underline{t}$ such that $\overline{x}^2 = \underline{y}^2 = (\overline{x}\ \underline{y})^9 = \overline{t}^2 = (\overline{x}\ \underline{t})^2 = (\underline{y}\ \underline{t})^2 = 1$ for all primes $p \leq 199$. Here p denotes the prime congruent to $\pm 1 \pmod 9$ and $f(\theta)$ denotes the polynomial $\theta^3 - 6\theta^3 + 9\theta - 1$.

P	Roots of $f(\theta)$	x	y	t
17	9	$z\to\frac{9z-3}{-z-9}$	$z\to\frac{z+12}{4z-2}$	$z\to\frac{-3}{z}$
	15			
	16	$z\to\frac{-3z+2}{-5z+3}$	$z\to\frac{z-5}{4z-2}$	$z\to\frac{-3}{z}$
		$z\to\frac{-z+7}{7z+1}$	$z\to\frac{4}{4z-1}$	$z\to\frac{-1}{z}$
19	5	$z\to\frac{13z+1}{z-13}$	$z\to\frac{z+4}{4z-2}$	$z\to\frac{-1}{z}$
	9	$z\to\frac{-2z-3}{3z+2}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
	11	$z\to\frac{9z-3}{-z-9}$	$z\to\frac{z+12}{4z-2}$	$z\to\frac{-3}{z}$
37	7	$z\to\frac{-2z-1}{z+2}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
	11	$z\to\frac{-3z+8}{4z+3}$	$z\to\frac{3z+14}{7z-4}$	$z\to\frac{-2}{z}$
	25	$z\to\frac{-13z+17}{-10z-13}$	$z\to\frac{6z-5}{16z-7}$	$z\to\frac{-2}{z}$
53	6	$z\to\frac{8z+2}{z-8}$	$z\to\frac{z+10}{5z-2}$	$z\to\frac{-2}{z}$
	16	$z\to\frac{4z+3}{3z-4}$	$z\to\frac{30}{30z+1}$	$z\to\frac{-1}{z}$
	37	$z\to\frac{-z+7}{30z+1}$	$z\to\frac{z+10}{5z-2}$	$z\to\frac{-2}{z}$
71	43	$z\to\frac{-17z-19}{19z+17}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
	48	$z\to\frac{27z-34}{34z-27}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
	57	$z\to\frac{z+12}{-12z-1}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
73	41	$z\to\frac{11z+7}{-7z-11}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
	50	$z\to\frac{-36z+42}{23z+36}$	$z\to\frac{2z+29}{35z-3}$	$z\to\frac{-5}{z}$
	61	$z\to\frac{-6z+5}{z+6}$	$z\to\frac{2z+29}{35z-3}$	$z\to\frac{-5}{z}$
89	50	$z\to\frac{-z-35}{35z+1}$	$z\to\frac{-1}{z-1}$	$z\to\frac{1}{z}$
	55	$z\to\frac{5z+9}{-3z-5}$	$z\to\frac{z-3}{z-2}$	$z\to\frac{3}{z}$
	79	$z\to\frac{8z+15}{-5z-8}$	$z\to\frac{z-3}{z-2}$	$z\to\frac{3}{z}$

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P	Roots of $f(0)$	x	y	t
107	49	$z\to\frac{-11z-29}{78z+11}$	$z\to\frac{z+5}{5z+52}$	$z\to\frac{-1}{z}$
	102	$z\to\frac{88z-24}{24z+19}$	$z\to\frac{-1}{z+106}$	$z\to\frac{-1}{z}$
109	25	$z\to\frac{-19z+40}{40z+19}$	$z\to\frac{33}{33z+108}$	$z\to\frac{-1}{z}$
	93	$z\to\frac{34z-53}{28z+75}$	$z\to\frac{33}{33z+108}$	$z\to\frac{-1}{z}$
	106	$z\to\frac{-13z+32}{16z+13}$	$z\to\frac{z+44}{22z+105}$	$z\to\frac{-2}{z}$
127	15	$z\to\frac{57z+68}{68z+57}$	$z\to\frac{z+39}{39z+125}$	$z\to\frac{-1}{z}$
	42	$z\to\frac{-55z+28}{28z+55}$	$z\to\frac{z+39}{39z+125}$	$z\to\frac{-1}{z}$
	76	$z\to\frac{51z+58}{69z+76}$	$z\to\frac{z-1}{z+126}$	$z\to\frac{1}{z}$
163	10	$z\to\frac{4z+31}{31z+159}$	$z\to\frac{z+46}{46z+161}$	$z\to\frac{-1}{z}$
	64	$z\to\frac{19z+10}{5z+144}$	$z\to\frac{2}{z+18}$	$z\to\frac{-2}{z}$
	95	$z\to\frac{61z+4}{4z+102}$	$z\to\frac{z+46}{46z+161}$	$z\to\frac{-1}{z}$
179	82	$z\to\frac{-15z+16}{16z+15}$	$z\to\frac{z+36}{36z+88}$	$z\to\frac{-1}{z}$
	135	$z\to\frac{29z+3}{3z+150}$	$z\to\frac{z+36}{36z+88}$	$z\to\frac{-1}{z}$
	147	$z\to\frac{67z+146}{33z+112}$	$z\to\frac{-1}{z+178}$	$z\to\frac{1}{z}$
181	106	$z\to\frac{80z+174}{87z+101}$	$z\to\frac{2z+158}{79z+178}$	$z\to\frac{-2}{z}$
	125	$z\to\frac{61z+21}{21z+120}$	$z\to\frac{19}{19z+180}$	$z\to\frac{-1}{z}$
	137	$z\to\frac{97z+143}{19z+84}$	$z\to\frac{2z+75}{53z+178}$	$z\to\frac{2}{z}$
197	36	$z\to\frac{44z+178}{19z+153}$	$z\to\frac{-1}{z+196}$	$z\to\frac{1}{z}$
	171	$z\to\frac{-4z-7}{-7z+4}$	$z\to\frac{-1}{z+196}$	$z\to\frac{1}{z}$
	193	$z\to\frac{-19z-23}{23z+19}$	$z\to\frac{-1}{z+196}$	$z\to\frac{1}{z}$
199	8	$z\to\frac{34z-13}{13z+165}$	$z\to\frac{-1}{z+198}$	$z\to\frac{1}{z}$

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P	Roots of $f(\theta)$	x	y	t
	36	$z \to \frac{-11z-91}{91z+11}$	$z \to \frac{-1}{z+198}$	$z \to \frac{1}{z}$
	161	$z \to \frac{-40z-3}{3z+159}$	$z \to \frac{-1}{z+198}$	$z \to \frac{1}{z}$

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