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ASSOCIATIVITY OF ($f_0, f_1, \ldots, f_k$)– POLYNOMIAL MODULE

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Abstract

Unless stated otherwise R will stand for a unitary ring, not necessarily associative, and M for a right unitary R-module, for which the associative property (i.e., $\alpha(ab) = (\alpha a)b$ for all α in M *and* a,b in R) is not assumed.

If R is associative, the ordinary polynomial ring $R[x]$ is associative and in obvious notation $M[x]$ is also associative as right $R[x]$-module, whenever M is associative.

In [2] the construction of an ($f_0, f_1, \ldots, f_k$) – extension ring $S = R[x; f_0, f_1, \ldots, f_k]$ of R was given, and its associativity was discussed. Even if R is associative, it is not necessary that S is associative. However, certain necessary and sufficient conditions were found in [2] in order that S be associative.
In [1] the construction of a polynomial module

$$N = M[x; f_0, f_1, \ldots, f_k]$$

of M was taken up and it was shown that N is a right S-module. If R and M are associative, then the problem of associativity of N arises. In this paper we intend to lay down conditions under which N *is* ensured to be associative. Indeed, these conditions look like those as in [1] and they are also studied in [3].

Keywords: Polynomial module.

1. Introduction

Throughout the paper R will stand for a unitary ring, not necessarily associative, and M for a right unitary module for which the associative property

$$\alpha(a\,b) = (\alpha\,a)b \quad \text{for all } \alpha \in M \qquad a, b \in R$$

is not assumed, unless stated otherwise.

In case R is asssociative it is well-known that ordinary polynomial ring $R[x]$ is associative and with obvious notation it is easy to check that $M[x]$ is also associative as right $R[x]$ -module whenever M is associative.

The construction of a left ($f_0, f_1, \ldots, f_k$)-extension

$$S = R[x; f_0, f_1, \ldots, f_k]$$

of R was given in [2] and among various properties , its associativity was dicussed. Even if R is associative S need not be associative (see [2, 2.7]). However, certain necessary and sufficient conditions for its associativity were laid down in [2].

In [1] the construction of ($f_0, f_1, \ldots, f_k$)-polynomial module

$$N = M[x; f_0, f_1, \ldots, f_k]$$

was taken up and it was shown that N is a right S-module , but no attempt was made to go through the associativity of N. Here in this paper we shall deal with the associativity of N and lay down some conditions under which it is associative.

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2. Polynomial Module

First of all we shall give the basic ingredients of N necessary for the development of associativity. Consider a fixed positive integer k. Using R^+ for the additive group of R, let us consider for each integer i an endomorphism of R^+, which is zero when i is other than $0, 1, \ldots, k$.

If n is a positive integer, then for each integer r, we define an endomorphism (n, r, f) by

$$(n, r, f)(c) = \sum f_{i_1} f_{i_2} \ldots f_{i_n}(c), \quad c \in R,$$

where the summation takes place over all $i_1, \ldots, i_n$ with their sum equal to r.
For simplicity, we write $i_1, \ldots, i_n$

$$(0, 0, f) = 1_{R^+},$$

and

$$(n, r, f)(c) = c_{n,r}.$$

Thus

$$c_{0,0} = c,$$
$$c_{0,r} = 0 \text{ in case } r \neq 0,$$
$$c_{n,r} = 0 \text{ when } r > nk \text{ or } r < 0,$$
$$c_{n,0} = (f_0)^n(c),$$
$$c_{1,r} = f_r(c),$$
$$c_{n,nk} = (f_k)^n(c).$$

It is clear that *for computation purpose it is enough to look for non-negative indices in* $f_{i_1} \ldots f_{i_n}$. Write, for each non-negative integer i,

$$R_i = R, \qquad M_i = M,$$

$$S = \coprod_0^\infty R_i, \quad N = \coprod_0^\infty M_i,$$

and for

$$\alpha = (\alpha_0, \ldots, \alpha_l, 0, \ldots) \in N,$$
$$b = (b_0, \ldots, b_m, 0, \ldots) \in S,$$

define αb by

$$(A_0, A_1, \ldots, A_u, \ldots),$$

where

$$A_u = \sum_{j=0}^{u} \sum_{i=0}^{l} \alpha_i \left(b_j\right)_{i,u-j} \qquad \ldots (A)$$

(A) makes sense, since

$$u > lk + n \Rightarrow u > ik + j \text{ for all } i, j$$
$$\Rightarrow u - j > ik \text{ for all } i, j$$
$$\Rightarrow A_u = 0.$$

Thus αb is an element of N and by [1, 2.2], we have

2.1 Proposition

With α, b, αb as above, S is a ring and N is a right S-module

If $h_M : M \to N$ is defined by

$$h_M(\alpha_0) = (\alpha_0, 0, \ldots)$$

for all α_0 in M, then it follows that

(i) h_R is a ring monomorphism,

(ii) N is a right R-module through h_R,

and

(iii) h_M is a right R-module homomorphism.

As such we identify M in N through h_M, R in S through h_R and write α_0, for $(\alpha_0, 0, \ldots)$.

2.2. Lemma

If $\alpha \in N$, $b_0 \in R$, $c \in S$, then for all u,

(i) $$(\alpha b_0)_u = \sum_{i=0}^{l} \alpha_i (b_0)_{i,u} ,$$

(ii) $$(b_0 c)_u = b_0 c_u .$$

Proof :

The first part follows immediately from (A).
Also putting $M = R$, $\alpha = b_0$ and replacing b by c in (A), we have

$$\begin{aligned}(b_0 c)_u &= \sum_{j=0}^{u} b_0 (c_j)_{0,u-j} \\ &= b_0 (c_u)_{0,0} \\ &= b_0 c_u . \end{aligned}$$

□

2.3 Lemma

If $\alpha \in N$ and $b \in S$ with $\alpha_i = 0$ for $i \neq l$ and $b_j = 0$ for $j \neq m$,

Then

$$(\alpha b)_u = \alpha_l (b_m)_{l,u-m} \quad for \quad u \geq m ,$$
$$(\alpha b)_u = 0 \quad for \quad u < m .$$

Proof :

In case $u \geq m$, the first part follows immediately by (A), as also the second part when $u < m$. □

An element x in an over ring of R is called a left f-inderterminate (see also [1], [2], [3] and [4]) over R, if for $c, b_0, \ldots, b_m \in R$

$$b_0 + b_1 x + \ldots + b_m x^m = 0$$

implies that each b_i is zero, and

$$xc = f_0(c) + f_1(c)x + \ldots + f_k(c)x^k .$$

The existence of such an x was established in [3] when

$$f_1(1) = 1 \in \ker f_i \quad (i \neq 1) \qquad \ldots\ldots (B)$$

It should be remarked that 2.1-2.3 hold even if R is not unitary, and the existence of x was carried over without R being unitary or fulfilling (B) in [2].

In [1] we had

2.4 Proposition

Suppose that R satisfies (B). Then there exists a left f-indeterminate over R such that each element of S (resp. N) *has a unique representation as a left polynomial of the form*

$$a_0 + a_1x + \ldots + a_lx^l \quad (resp. \quad \alpha_0 + \alpha_1x + \ldots + \alpha_lx^l)$$

with each a_i in R (resp. α_i *in M*), *and as such we write*

$$S = R[x; f], \qquad N = M[x; f].$$

we call S (resp. N) left ($f_0, f_1, \ldots, f_k$)-polynomial or left f-polynomial ring (resp. module).

3. Associativity of N

In view of 2.3, we have

3.1 Lemma

For $\alpha_0, \alpha_l \in M, \quad b_m \in R,$

(i) $$(\alpha_l x_l)(b_m x_m) = \sum_{u=m}^{lk+m} \{\alpha_l (b_m)_{l,u-m}\} x^u$$

$$= \sum_{u=0}^{lk} \{\alpha_l (b_m)_{l,u}\} x^{u+m}$$

(ii) $\alpha_0 (b_m x^m) = (\alpha_0 b_m) x^m .$

It is easy to show that $I_{n,n} = 1$ and $I_{n,r} = 0$ *where* $r \neq n$.

3.2 Lemma

If $\alpha_l, \alpha_l \in M, \ b_0, b_m \in R$, then

(i) $(\alpha_l x^l) x^m = \alpha_l x^{l+m},$

(ii) $\{(\alpha_l x) b_m\} x^m = (\alpha_l x)(b_m x^m),$

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and (iii) $(\alpha_1 x)\, b_0 = \alpha_1 (x b_0)$.

Proof :

By 3.1 (i),

$$(\alpha_1 x^l) x^m = \sum_{u=m}^{lk+m} (\alpha_1\ l_{l,u-m})\, x^u = \alpha_1 x^{l+m},$$

$$\{(\alpha_1\ x) b_m\} x^m = \sum_{u=m}^{k} \left[\{\alpha_1\ (b_m)_{l,u}\} x^u\right] x^m,$$

$$= \sum_{u=0}^{k} \{\alpha_1\ f_u(b_m)\}\, x^{u+m}$$

$$= (\alpha_1\ x)(b_m x^m),$$

and

$$(\alpha_1\ x)\, b_0 = \sum_{u=0}^{k} \{\alpha_1\ f_u(b_0)\} x^u$$

$$= \alpha_1 \sum_{u=0}^{k} f_u(b_0) x^u .$$

$$= \alpha_1 (x b_0).$$

3.3 Lemma

If M is associative and $\alpha_0 \in M$, $b_0, c_0 \in R$, $b, c \in S$, then

(i) $\alpha_0 \{(b_0 x^m)(c_0 x^n)\} = \{\alpha_0 (b_0 x^m)\}(c_0 x^n)$,

(ii) $\alpha_0 (bc) = (\alpha_0 b) c$.

Proof :

(i) By 3.1, associativity of M, and distributive property, we get

$$\alpha_0 \{(b_0 x^m)(c_0 x^n)\} = \alpha_0 \sum_{u=n}^{mk+n} \{b_0 (c_0)_{m,u-n}\} x^u$$

$$= \sum_{u=n}^{mk+n} [\alpha_0 \{b_0 (c_0)_{m,u-n}\}] x^u$$

$$= \sum_{u=n}^{mk+n} \{(\alpha_0 b_0)(c_0)_{m,u-n}\} x^u$$

$$= \{(\alpha_0\ b_0) x^m\}(c_0 x^n)$$

$$= \{\alpha_0 (b_0 x^m)\}(c_0 x^n).$$

(ii) $$\alpha_0 (bc) = \alpha_0 \left\{ \left(\sum_i b_i x^i \right) \left(\sum_j c_j x^i \right) \right\}$$

$$= \alpha_0 \sum_{i,j} (b_i x^i)(c_j x^j)$$

$$= \sum_{i,j} \{\alpha_0 (b_i x^i)\}(c_j x^j)$$

$$= \left\{ \sum_i \alpha_0 (b_i x^i) \right\} \sum_j c_j x^j$$

$$= \left\{ \alpha_0 \sum_i b_i x^i \right\} c$$

$$= (\alpha_0 b) c. \quad \square$$

For $\alpha \in M$, $b, c \in R$, $m \geq 0$, write

$$\lambda = \alpha \sum_{u=0}^{k} f_u (b)\, c_{u+m,\, w}.$$

We now prove

3.4 Proposition

The following conditions are equivalent:

1. N is associative as right S- module,
2. (i) M is associative,

   (ii) $\lambda = 0$ in case $w > (m+1)k$, and

   (iii) $\lambda = \alpha \sum_{r=0}^{w} f_{w-r}(b\, c_{m,r})$ for $0 \leq w \leq (m+1)k$

   for all α, b, c, m, w.

Proof:

By 3.1 and 3.2, we have

$$(\alpha x)\{(bx^m)(cx^n)\} = (\alpha x) \sum_{u=n}^{mk+n} (bc_{m,\,u-n}) x^u$$

$$= \sum_u \{(\alpha x)(bc_{m,\,u-n})\} x^u$$

$$= \sum_u \left[\sum_{v=0}^{k} \{\alpha f_v (bc_{m,u-n})\} x^v \right] x^u$$

$$= \sum_v \{\alpha f_v (bc_{m,o})\} x^{v+n}$$

$$+ \sum_v \{\alpha f_v (bc_{m,1})\} x^{v+n+1} + \ldots + \sum_v \{\alpha f_v (bc_{m,r})\} x^{v+n+r} + \ldots$$

$$+ \sum_v \{\alpha f_v (bc_{m,nk})\} x^{v+n+mk}$$

$$= \{\alpha f_0 (bc_{m,o})\}x^n + \alpha\{f_1(bc_{m,o}) + f_o(bc_{m,1})\}x^{n+1} + \ldots$$
$$+ \alpha\{f_r(bc_{m,o}) + f_{r-1}(bc_{m,1}) + \ldots + f_o(bc_{m,r})\}x^{n+r} + \ldots$$
$$+ \alpha\{f_{mk+k}(bc_{m,o}) + \ldots f_0(bc_{m,mk+k})\}x^{n+mk+k}$$
$$= \sum_{r=0}^{mk+k}\left\{\sum_{t=0}^{r} f_{r-t}(bc_{m,t})\right\}x^{n+r}.$$

Thus

$$(\alpha x)\{(bx^m)(cx^n)\} = \sum_{r=0}^{mk+k}\left\{\sum_{t=0}^{r}\alpha f_{r-t}(bc_{m,t})\right\}x^{n+r} \qquad \ldots(C).$$

Again

$$\{(ax)(bx^m)\}(cx^n) = \left\{\sum_{u=m}^{k+m}(\alpha b_{1,u-m})x^u\right\}(cx^n)$$
$$= \left[\sum_{v=0}^{k}\{\alpha f_v(b)\}x^{v+m}\right](cx^n)$$
$$= \sum_{v}\sum_{w=n}^{(v+m)k+n}[\{\alpha f_v(b)\}c_{v+m,w-n}]x^w$$
$$= \sum_{w=n}^{mk+n}[\{\alpha f_o(b)\}c_{m,w-n}]x^w + \ldots + \sum_{w=n}^{(v+m)k+n}[\{\alpha f_v(b)\}c_{v+m,w-n}]x^w + \ldots$$
$$+ \sum_{w=n}^{(k+m)k+n}[\{\alpha f_k(b)\}c_{k+m,w-n}]x^w$$
$$= \{(\alpha f_o(b))c_{m,o} + (\alpha f_1(b))c_{1+m,o} + \ldots + (\alpha f_k(b))c_{k+m,o}\}x^n + \ldots$$
$$+ \{(\alpha f_0(b))c_{m,t} + (\alpha f_1(b))_{1+m,t} + \ldots + (\alpha f_k(b))c_{k+m,t}\}x^{n+t} + \ldots.$$
$$+ \{(\alpha f_o(b))c_{m,(k+m)k} + (\alpha f_1(b))c_{1+m,(k+m)k} + \ldots + (\alpha f_k(b))c_{k+m,(k+m)k}\}x^{n+(k+m)}$$

So,

$$(\alpha x)\{(bx^m)(cx^n)\} = \sum_{r=0}^{(k+m)k}\left\{\sum_{v=0}^{k}(\alpha f_v(b)\,c_{v+m,r})\right\}x^{n+r} \qquad \ldots(D)$$

$(i)'$ Suppose that (1) holds. Then, for α *in* M, $b, c \in R$, we have

$$\alpha(bc) = \alpha\{(bx^0)(cx^0)\} = \{\alpha(bx^0)\}(cx^0) = (\alpha b)c,$$

so that M is associative. Now using (C) and (D),

$$(\alpha x)\{(bx^m)(cx^n)\} = \{(\alpha x)(bx^m)\}(cx^n)$$
$$\Rightarrow \sum_{r=0}^{mk+k}\left\{\sum_{t=0}^{r}\alpha f_{r-t}(bc_{m,t})\right\}x^{n+r} = \sum_{r=0}^{mk+k^2}\left\{\sum_{v=0}^{k}\alpha f_v(b)c_{v+m,r}\right\}x^{n+r},$$

which implies (*ii*), (*iii*) of (2).

$(ii)'$ Conversely, suppose that (2) holds. Then by (C) and (D),

$$(\alpha x)\{(bx^m)(cx^n)\} = \sum_{r=0}^{mk+k}\left\{\sum_{t=0}^{r}\{\alpha f_{r-t}(bc_{m,t})\}x^{n+r}\right\}$$

$$= \sum_{r=0}^{mk+k}\left\{\sum_{v=0}^{k}\{\alpha f_v(b)c_{v+m,r}\}x^{n+r} + \sum_{r=mk+k+1}^{mk+k^2}\left\{\sum_{v=0}^{k}\{\alpha f_v(b)c_{v+m,r}\}\right\}x^{n+r}\right\}$$

(the second term is not required when $k=1$).

$$= \sum_{r=0}^{mk+k^2}\left\{\sum_{v=0}^{k}\alpha f_v(b)c_{v+m,r}\right\}x^{n+r}.$$

$$= \{(\alpha x)(bx^m)\}(cx^n).$$

Thus

$$(\alpha x)\{(bx^m)(cx^n)\} = \{(\alpha x)(bx^m)\}(cx^n), \qquad \text{... (E)}$$

for $\alpha \in M$, $b, c \in R$ and m, n non-negative integers.

Consider $b(x) = \sum_{j=0}^{m} bjx^j$ in S. Then using the distributive law and (E), we get

$$(\alpha x)\{b(x).(cx^n)\} = (\alpha x)\left\{\sum_j (b_j x^j)(cx^n)\right\}$$

$$= \sum_j (\alpha x)\{(b_j x^j)(cx^n)\}$$

$$= \sum_j \{(\alpha x)(b_j x^j)\}(cx^n)$$

$$= \left\{\sum_j (\alpha x)(b_j x^j)\right\}(cx^n)$$

$$= \left\{(\alpha x)\sum_j b_j x^j\right\}(cx^n)$$

$$= \{(\alpha x).b(x)\}(cx^n).$$

So,

$$(\alpha x)\{b(x).(cx^n)\} = \{(\alpha x).b(x)\}(cx^n)$$

If $c(x) = \sum_{k=1}^{n} c_k x^k$ in S, then by this

$$(\alpha x)\{b(x).c(x)\} = (\alpha x)\left\{\sum_k b(x)(c_k x^k)\right\}$$

$$= \sum_k (\alpha x)\{b(x).(c_k x^k)\}$$

$$= \sum_k \{(\alpha x) b(x)\}(c_k x^k) =$$
$$= \{(\alpha x)..b(x)\} \sum_k c_k x^k$$

So,

$$(\alpha x)\{b(x).c(x)\} = \{(\alpha x).b(x)\}c(x). \qquad \ldots (G)$$

Let $\alpha(x) = \sum_0^l \alpha_i x^i$ belong to N and assume that

$$(\alpha x^{l-1})\{b(x).c(x)\} = \{(\alpha x^{l-1}).b(x)\}c(x).$$

Then

$$\begin{aligned}(\alpha x^l)\{b(x).c(x)\} &= \{(\alpha x^{l-1})x\}\{b(x).c(x)\} \\ &= (\alpha x^{l-1})[x\{b(x).c(x)\}] \\ &= (\alpha x^{l-1})[\{xb(x)\}c(x)] \\ &= \{(\alpha x^{l-1})(xb(x))\}c(x), \\ &= \{(\alpha x^l)b(x)\}c(x),\end{aligned}$$

so that by induction, for all l and $\alpha \in M$,

$$\{(\alpha x^l)b(x)\}c(x) = (\alpha x^l)\{b(x).c(x)\}.$$

Finally,

$$\begin{aligned}\alpha(x)\{b(x).c(x)\} &= \sum_i \alpha_i x^i \{b(x).c(x)\}. \\ &= \sum_i (\alpha_i x^i)\{b(x).c(x)\} \\ &= \sum_i \{(\alpha_i x_i)b(x)\}c(x) \\ &= \left\{\sum_i (\alpha_i x^i)b(x)\right\}c(x) \\ &= \{\alpha(x)b(x)\}c(x).\end{aligned}$$

Hence N is associative. □

3.5 Corollary

If, for $m \geq 0, b, c \in R$ and $\mu(m,b,c) = \sum_{u=0}^{k} f_u(b)c_{u+m,w}$, then N is associative in case

(i) M is associative,

(ii) $\mu(m,b,c) = \sum_{r=0}^{w} f_{w-r}(bc_{m,r})$ whenever $0 \leq w \leq (m+1)k$, and

(iii) $\mu(m,b,c) = 0$ otherwise, for all m, b, c.

Proof:

Follows immediately, since (2) of 3.4 is satisfied for all α in M. □

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Since R is unitary, immediately from 3.4, we have 3.6 Corollary

3.6 Corollary

Suppose that R is associative. Then S is associative if and only if

$$\text{(i)} \quad \sum_{r=0}^{w} f_{w-r}(ab_{j,r}) = \sum_{i=0}^{w} f_i(a)b_{i+j,w}, \text{ with } 0 \le w \le (j+1)k, \text{ and}$$

$$\text{(ii)} \quad \sum_{i=0}^{k} f_i(a)b_{i+j,w} = 0 \text{ with } w > (j+1)k$$

for all non-negative integers j and a, b in R.

3.7 Corollary

If M, S are associative, then N is associative.

Proof:

Follows from 3.6 and 3.5.

Throughout the above discussion we have used extensively the conditions

(i) R has identity element,
(ii) M is unitary,
(iii) $f_1(1) = 1$,
(iv) $f_i(1) = 0$ in case $i \neq 1$.

In fact the reader can observe that some of the results remain valid without these. Indeed, we can obtain all the consequences, if R or M fail to fulfill any collection of such assumptions. A clear indication of the procedure is given in [1], where 2.1 is proved (see [1,3.2]) quite generally. Concluding we state that all the above results can be carried over in the general set up, (see [4]).

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