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FUZZY RELATIONS AND FUZZY SUBALGEBRAS OF BCI- ALGEBRAS

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Abstract

In this paper, we define and study fuzzy relations, fuzzy subalgebras and fuzzy H. S-subalgebras of a BCI-algebra X ×X.

Keywords: **Fuzzy relation, strongest fuzzy relation, fuzzy H-subalgebra, fuzzy S-subalogebra, weakest fuzzy subset and fuzzy K-subset.**

Introduction

L.A.Zadeh [10], introduced the concept of fuzzy sets in 1965. After that the researchers applied this concept to other mathematical disciplines such as groups and topology [9],[3]. Now-a-days, the theory of fuzzy sets has developed in many directions and is finding applications in a variety of fields. In 1991, X.Ougen [8], applied it to BCK- algebras. In 1993, B.Ahmed [1], and Y.B.Jun [5], applied it to BCI-algebras. In this paper, we study fuzzy relations, fuzzy sub algebras and fuzzy H,S-subalgebras of a BCI- algebra.

Recall that an algebra. (X,*,0) of type (2,0) is called a BCI-algebra, if it satisfies the following conditions:

(1) $(x * y) * (x* z) \leq z * y$,
(2) $x * (x* y) \leq y$,
(3) $x \leq x$,
(4) $x \leq y$ and $y \leq x$ imply $x = y$,
(5) $x \leq 0$ and imply $x = 0$,

where $x \leq y$ mean $x^*= 0$.

The following holds in a BCI-algebra

(6) $x*0 = x$, [4].

A BCI-algebra X is called associative if for all $x\in X$.

(7) $0^*x=x$.

Preliminaries

We briefly review some definitions and results. We recall [5],[1] that a fuzzy set in a BCI-algebra X is a mapping μ from X into [0,1]. For any fuzzy set μ in X and $t \in [0,1]$, the set $\mu_t = \{ x \in X, \mid \mu (x) \geq t\}$ is called a level subset of μ. For any fuzzy sets μ and σ in X, $\mu = \sigma$ iff $\mu (x) = \sigma (x)$ for all $x \in X$, and $\mu \subseteq \sigma$ iff $\mu (x) \leq \sigma (x)$, for all $x \in X$, [7].

Let X be a BCI-algebra. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy subalgebra, [1], of X, if for all , $x,y \in X$,

$$\mu (x*y) \geq \mu (x) \wedge \mu (y),$$

where $\wedge$ denotes for minimum

It is easy to see that $\mu (0) \geq \mu (x)$, for all $x \in X$.

Following Bhattacharya and Mukharjee [2], Jun and Meng [6], defined and studied the notions of fuzzy relations, strongest fuzzy relations in a BCI-algebra. In [7], H.M.Khalid and B. Ahmed studied some further interesting properties.

Definition 1 [6]. Let X be a BCI-algebra. A fuzzy relation μ on X is a fuzzy subset of $X \times X$, that is, a mapping $\mu: X \times X \rightarrow [0,1]$.

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Defination 2 [6]. Let μ be a fuzzy relation on a BCI-algebra X and σ fuzzy subset of X. Then μ is a fuzzy relation on σ, if for any $x,y \in X$,

$\mu_\sigma (x,y) = \sigma (x) \wedge \sigma (y)$,

Theorem 1 [6]. Let σ be a fuzzy subset of a BCI-algebra X and μ a fuzzy relation on σ. Then level subset μ_t, $t \in [0,1]$ is a fuzzy relation on σ_t.

Defination 3 [6]. If σ is fuzzy subset of a BCI-algebra X, then a fuzzy relation μ on σ, denoted u_σ, given by

$u_\sigma (x,y) = \sigma (x) \wedge \sigma (y)$,

is called the strongest fuzzy relation.

The following theorem gives a complete list of the level subsets of u_σ in terms of level subsets of σ:

Theroem 2. For a given fuzzy subset σ of a BCI-algebra X, let μ σ be the strongest relation on X, then for $t \in [0,1]$, we have

$(u_\sigma)_t = \sigma_t \times \sigma_t$

Proof. Let $(x,y) \in (u_\sigma)_t$. Then

$u_\sigma (x,y) \geq t, t \in [0,1]$. This gives $\sigma (x) \wedge \sigma (y) \geq t, t \in 0,1$. Obviously, $(x,y) \in \sigma_t \times \sigma_t$.

Conversely, let $(x,y) \in \sigma_t \times \sigma_t$. Then $\sigma (x) \geq t$, $\sigma (y) \geq t$, which imply $\sigma (x) \wedge \sigma (y) \geq t$. This gives $\mu_\sigma (x,y) \geq t$ or $(x,y) \in (\mu_\sigma)_t$. This completes the proof.

Next, we define,

Defination 4. Let μ be a fuzzy relation on X. Then μ is called a fuzzy subalgebra of $X \times X$, if for all (x_1,y_1), $(x_2,y_2) \in X \times X$, we have
$\mu ((x_1,y_1) * (x_2,y_2)) \geq \mu (x_1,y_1) \wedge \mu (x_2,y_2)$.

Similarly, we have:

Defination 5. Let μ be a fuzzy relation on a BCI-algebra X and σ a fuzzy subset of X. Then the strongest fuzzy relation μ_σ is called a fuzzy subalgebra of $X \times X$ if for all (x_1,y_1), $(x_2,y_2) \in X \times X$, $\mu_\sigma ((x_1,y_1)^* ((x_2,y_2)) \geq \mu_\sigma (x_1,y_1) \wedge \mu_\sigma (x_2,y_2)$.

Put $x_1 = x_2 = x$ and $y_1 = y_2 = y$, then for all $(x, y) \in X \times X$.

$\mu_\sigma (0, 0) \geq \mu_\sigma (x,y)$ ..... (1)

Lemma 1. Let σ be a fuzzy subset of a BCI-algebra X. If μ_σ is a fuzzy subalgebra of $X \times X$, then $\sigma (X) \leq \sigma (0)$, for all $x \in X$.

Proof. From (1), $\mu_\sigma (0, 0) \geq \mu_\sigma (x,y)$. This gives $\sigma (0) \wedge \sigma (0) \geq \sigma (x) \wedge \sigma (y)$. Puting $x = y$, we have $\sigma (0) \geq \sigma (x)$. This completes the proof.

Using lemma 1, we characterize fuzzy subalgebra μ_σ in terms of a fuzzy subalgebra σ as:

Theorem 3. Let σ be a fuzzy subset of a BCI-algebra X. Then σ is a fuzzy subalgebra of X iff μ_σ is a fuzzy subalgebra of $X \times X$.

Proof. Suppose σ is a fuzzy subalgebra of X. Then

$\sigma (x_1 * x_2) \geq \sigma (x_1) \wedge \sigma (x_2)$ and $\sigma (y_1 * y_2) \geq \sigma (y_1) \wedge \sigma (y_2)$.

Consequently, we have

$\sigma (x_1 * x_2) \wedge \sigma (y_1 * y_2) \geq (\sigma (x_1 \wedge \sigma (x_2)) \wedge (\sigma (y_1) \wedge \sigma (y_2))$ ..... (2)

Now we assume that

$\sigma (x_1) \leq \sigma (x_2)$.....(3)

Here we discuss two cases:

Case 1. Suppose $\sigma(y_1) < \sigma(y_2)$. Then from (2) and (3),

$\sigma(x_1 * x_2) \wedge \sigma(y_1 * y_2) \geq \sigma(x_1) \wedge \sigma(y_1)$

$\geq \sigma(x_1) \wedge (\sigma(x_2) \wedge \sigma(y_2)$ or

$\sigma(x_1 * x_2) \wedge \sigma(y_1 * y_2) \geq (\sigma(x_1) \wedge (\sigma(y_1))$

$\wedge (\sigma(x_2) \wedge \sigma(y_2))$ or

$\mu_\sigma(x_1 * x_2, y_1 * y_2) \geq \mu_\sigma(x_1, y_1) \wedge \mu_\sigma(x_2, y_2)$

This shows that μ_σ is a fuzzy subalgebra $X \times X$.

Case 2. Suppose $\sigma(y_1) > \sigma(y_2)$. Then from (2) and (3),

$\sigma(x_1 * x_2) \wedge \sigma(y_1 * y_2) \geq \sigma(x_1) \wedge (\sigma(y_2)$. ..... (4)

But since

$\sigma(x_1) \geq \sigma(x_1) \wedge \sigma(y_1$ and $\geq \sigma(x_2)$, we have $(\sigma(x_1) \wedge \sigma(y_2)) \leq \sigma(x_1) \wedge \sigma(y_2)$ or
$\mu_\sigma(x_1, y_1) \geq \mu_\sigma \wedge (x_2, y_2)$
$\leq \sigma(x_1) \wedge \sigma(y_2)$.....(5)

Then from (4) and (5), we obtain

$\mu_\sigma(x_1 * x_2, y_1 * y_2) \geq \mu_\sigma(x_1, y_1) \wedge \mu_\sigma(x_2, y_2)$ or

$\mu_\sigma(x_1, y_1) * (x_2, y_2)) \geq \mu_\sigma(x_1, y_1) \wedge \mu_\sigma(x_2, y_2)$.

This completes the necessity.

Conversely, assume that μ_σ is a fuzzy subalgebra of $X \times X$. We shall prove that for all $x, y \in X$,

$\sigma(x * y) \geq \sigma(x) \wedge \sigma(y)$. ..... (6)

Since μ_σ is a fuzzy subalgebra of $X \times X$, we have

$\mu_\sigma(x_1 * x_2, y_1 * y_2) = \mu_\sigma(x_1, y_1) * (x_2, y_2)$.

$\geq \mu_\sigma(x_1, y_1) \mu_\sigma(x_2, y_2) = (\sigma(x_1) \wedge \sigma(y_1) \wedge (\sigma(x_2) \wedge \sigma(y_2))$.

or $\sigma(x_1 * x_2) \wedge \sigma(y_1 * y_2) \geq (\sigma(x_1) \wedge \sigma(y_1)) \wedge (\sigma(x_2) \wedge \sigma(y_2))$. .....(7)

Now we put $x_1 = x_2 = y$, $y_1 = 0 = y_2$ in (7), we get

$\sigma(x * y) \geq (\sigma(x) \wedge \sigma(0)) \wedge (\sigma(y) \wedge \sigma(0))$.....(8)

Using lemma 1, (8) becomes

$\sigma(x * y) \geq \sigma(x) \wedge \sigma(y)$.

This proves that σ is a fuzzy subalgebra of X. This completes the proof.

Next, we define :

Definition 6. Let σ be a fuzzy subset of a BCI-algebra X and μ a fuzzy relation on σ. Then the fuzzy subset of X, denoted σ_μ, defined by

$$\sigma_\mu(x) = \sup_{y \in X} \{\mu(x,y) \vee \mu(y, x)\}$$

is called the weakest fuzzy subset of X, where $\vee$ stands for maximum.
Now, we define fuzzy H-subalgebra as:

Definition 7. Let σ be a fuzzy subset of a BCI-algebra X. Then the fuzzy subalgebra

μ of $X \times X$ is called fuzzy H-subalgebra of X, if for all $x, y \in X$,

$\mu(x,z) \leq \mu(y, 0)$.

Using the concept of fuzzy H-subalgebra, we show that the weakest subset σ_μ is a fuzzy subalgebra in a BCI-algebra X.

Theorem 4. If μ is a fuzzy H-subalgebra of a BCI-algebra $X \times X$, then the weakest fuzzy subset σ_μ of X is a fuzzy subalgebra of a BCI-algebra X.

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Proof. Let be a BCI-algebra and x, y ∈ X. We prove that

$\sigma_\mu(x * y) \geq \sigma_\mu(x) \wedge \sigma_\mu(y)$ .....(1)

Here we may assume that $\sigma_\mu(x) < \sigma_\mu(y)$. Then (∈) becomes $\sigma_\mu(x * y) \geq \sigma_\mu(x)$, that is, we show that

$$\sup_{y \in X} \{\mu(x * y,) \vee \mu(z, x * y)\} \geq \sup_{z \in X} \{\mu(x,) \vee \mu(,x)\}$$

Since μ is a fuzzy subalgebra, we have

$\mu(x * y, z) = \mu(x, y, z * 0)$

$= \mu((x,z) * (y,0))$

$\geq \mu(x, z) \wedge \mu(y,0)$

or $\mu(x * y, z) \geq \mu(x, z) \wedge \mu(y,0)$.

Since μ is a fuzzy H-subalgebra, then we have

$\mu(x * y, z) \geq \mu(x, z)$ ... (2)

Similarly, we can get

$\mu(z, y * z) \geq \mu(z,x)$. .... (3)

(2) and (3) give

$\mu(x * y, z) \vee \mu(z,x, * y) \geq \mu(z,x)$ which gives

$$\sup_{y \in X} \{\mu(x * y, z) \vee \mu(z,x, * y)\} \geq \sup_{z \in X} \{\mu(x,z) \vee \mu(z,x)\}$$

or $\sigma_\mu(x * y) \geq \sigma_\mu(x) \geq \sigma_\mu(x) \wedge \sigma_\mu(y)$.

This proves that σ_μ is a fuzzy subalgebra of X, Hence the proof.

Definition 8. Let σ be a fuzzy subset of a BCI-algebra X. Then the fuzzy subalgebra

μ of X× X is called fuzzy S-subalgebra of X if, for all x,y, z ∈ X,

$\mu(x, z) \leq \mu(0,y)$.

Definition 9. Let μ be a relation on X and x ∈ X. Then we define fuzzy K- subset of

X,denoted σ_x, as:

$\sigma_x(y) = \mu(x,y)$, for all y ∈ X.

Finally, we show that the fuzzy K-subset σ_x of X is a fuzzy subalgebra of X, provided μ is a fuzzy S-subalgebra of X × X.

Theorem 5. Let X be an associative BCI-algebra. If μ is a fuzzy S-subalgebra of a BCI algebra X× X, then for any x ∈ X, the fuzzy K-subset σ_x is a fuzzy subalgebra of X.

Proof. For $y_1, y_2 \in X$, we have

$\sigma_x(y_1 * y_2) = \mu(x,y_1,y_2)$, (def. 9)

$= \mu(x * 0, y_1 * y_2)$

$\geq \mu(x, y_1) * y_2) \wedge \mu(0, y_2)$.

or $\sigma_x(y_1 * y_2) \geq \mu(x,y_1,) \wedge \mu(0,y_2)$.

Since μ is fuzzy S-subalgebra of X × X, therefore we have

$\sigma_x(y_1 * y_2) \geq \mu(x,y_1) = \sigma_x(y_1)$, (def. 9)

or $\sigma_x(y_1 * y_2) \geq \sigma_x(y_1)$. Next, consider

$\sigma_x(y_1 * y_2) = \mu(x, y_1 * y_2)$

$= \mu(0 * x, y_1 * y_2)$ (X is associative)

$= \mu(0, y_1) * (x,y_2)) \geq \mu(0, y_1) \wedge \mu(x,y_2)$

$= \mu(x, y_2) = \sigma_x(y_2)$,

which gives $\sigma_x (y_1 * y_2) \geq \sigma_x (y_2)$. Consequently,

$\sigma_x (y_1 * y_2) \geq \sigma_x (y_1) \wedge \sigma_x (y_2)$.

This proves that the fuzzy K-subset σ_x is a fuzzy subalgebra of X.

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